The Stochastic Cauchy Problem driven by a cylindrical Levy Process

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A thesis presented for the degree of

Doctor of Philosophy in Applied Mathematics

May 2018

Abstract

This thesis deals with the study of linear stochastic partial differential equations driven by cylindrical Lévy processes. Cylindrical Lévy processes were recently introduced as a natural generalisation of cylindrical Wiener processes. A good part of the thesis is to obtain necessary and sufficient conditions for the existence of a weak and mild solution of the abstract stochastic Cauchy problem driven by a cylindrical Lévy process in a separable Hilbert space. The methods employed are to use the techniques of strongly continuous semi-group theory and the recently developed stochastic integration theory for integrating deterministic functions with respect to cylindrical Lévy processes. These techniques are first employed to prove a stochastic version of the Fubini theorem to stochastic integrals with respect to cylindrical Lévy processes, which in turn is used to prove the existence of the weak solution. Some further theoretical properties of the solution such as the Markov property and stochastic contnuity are derived. The necessary and sufficient conditions for the existence of the invariant measure for the stochastic Cauchy problem are obtained when the semigroup is stable. For specific examples including the stochastic heat equation driven by cylindrical Lévy process, necessary and sufficient conditions as generalisation of the log moment condition for genuine Lévy processes are obtained, which are satisfied by many examples of cylindrical Lévy processes.

Acknowledgements

First and foremost, I want to thank my supervisor Prof Markus Riedle for his constant support and motivation throughout this journey of my PhD. He has been instrumental in not only guiding me along the path that led to this final destination, but also in making the journey fruitful and enjoyable, even in the periods when the path was not so smooth. His constant encouragement, patience, understanding and supportive nature, and his excitement for mathematics is worth applauding. I very much enjoyed our research seminars and the weekly lunches together.

I also thank Dr Gechun Liang, my second supervisor in the first three years of my PhD. The combined seminars with him in my first year were very enlightening and I very much enjoyed giving seminars in our joint research group and his encouragement in those days was very energising.

I specially mention Tomasz Kosmala, my PhD colleague and friend who supported me throughout the period of PhD both on a personal and academic level. Our mutual discussions on mathematical problems were very enlightening and his simple and caring attitude is appreciable. He deserves special thanks for reading many versions of my thesis and for providing some very useful comments and suggestions.

I specially thank Matthew Griffiths for proofreading most part of my thesis and providing useful comments as well as for some useful discussions and Dr Jian Liang Zhai for many interesting discussions during his stay as a postdoc at King's.

I also express my sincere thanks to Prof Erika Hausenblas for inviting me to visit Montanuniversität, Leoben and for organising my talk in her research group, to Prof Jan van

Neerven for clarifying some of my doubts and for some enriching discussions in the symposium at Durham, to Prof David Applebaum for his encouragement and for suggesting some papers to me while I visited Sheffield for a conference.

I express my gratitude to Department of Mathematics at King's College for providing full funding for PhD, to my home institution Rajdhani College, University of Delhi, India for giving me the study leave for PhD and to the donors of the Dr Doris Chen Mathematics scholarship for financial support. I am honoured to be the first recipient of this scholarship and express my sincere thanks to Prof William Chen for establishing this scholarship and for his motivation and encouragement for a good future career in mathematics and for kindly sending me his book.

I mention some of my fellow PhD students: Asuka Kumon and Kwok-Wing Tsoi for creating a support system and giving some useful suggestions on writing style, and my very dear and supportive friends Shubham Pandey, Sergio Maffra, Senbao Jiang and Konul Mustafayeva.

Special thanks to my colleagues in University of Delhi, Dr R. D. Sarma and Prof C. S. Lalitha for their constant encouragement and concerns for my well-being.

I will not be doing justice if I forget all my students in my tutorial classes at King's who nominated me for the Outstanding Teaching Assistant award consecutively for all four years of my PhD and which enabled me to get the Outstanding Teaching Assistant Runner's-up prize twice.

Finally, I thank my parents for their love and affection and also other members of my family.

Contents

Chapter 1

Introduction

Stochastic partial differential equations (SPDEs) can be used to model many situations and natural phenomena occurring in diverse fields such as physics, life sciences, economics and finance (see the introductory Chapter in [20] for many examples). For more than half a century, the standard model for the driving noise for random perturbation in SPDEs has been the cylindrical Brownian motion and the theory is well established than the non-Gaussian case (e.g. see [18], [19], [20] for Hilbert spaces and [54], [55],[58] for Banach spaces). The first paper to use a Hilbert space-valued Lévy process (which will be also called a genuine Lévy process in this thesis) as a driving noise to study SPDEs was by Chojnowska-Mikhalik [16] in 1987. Recently, after a gap of some period, there has been a lot of interest in studying SPDEs driven by noises more general than the genuine Lévy process as analogues of cylindrical Brownian motion in the non-Gaussian setting.

This thesis deals with the following linear stochastic evolution equation with additive noise, or equivalently the stochastic Cauchy problem (SCP):

$$\mathrm{d}Y(t) = AY(t)\,\mathrm{d}t + B\,\mathrm{d}L(t) \qquad \text{for all } t \in [0, T]. \tag{1.0.1}$$

Here, B is a linear bounded operator from a separable Hilbert space U to a separable Hilbert space V and A is the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ in V. The novelty is in the driving noise L, which is assumed to be a cylindrical Lévy process introduced recently by Applebaum and Riedle (2010) in [6] by following the cylindrical approach extending the definition of a finite-dimensional Lévy process to infinite dimensions. It can be considered as a natural generalisation of the cylindrical Wiener process and is defined using the rich theory of cylindrical measures and cylindrical random variables. Apart from cylindrical Wiener processes and genuine Lévy processes as examples, it also includes many noises considered in the literature e.g. subordinated Wiener processes in [14], cylindrical α-stable noises in [41], canonical α-stable noises in [42]. Only for specific examples of cylindrical Lévy processes, the stochastic Cauchy problem has been considered in the literature. However, in this work we develop a complete theory of existence of a weak and equivalently mild solution of (1.0.1) in the most general setting which unifies the theory for all examples mentioned above. We are able to obtain some fundamental properties of the solution, which were previously either not available for these examples or were proved with arguments specific to the example under consideration.

A sound basis for the theory of linear equations (1.0.1) is needed to study the more general semi-linear equations of the type

$$\mathrm{d}Y(t) = (AY(t) + F(Y(t)))\,\mathrm{d}t + B\,\mathrm{d}L(t) \qquad \text{for all } t \in [0,T], \tag{1.0.2}$$

where F is some non-linear operator in V. Although the semi-linear equations are even more important (for instance, in terms of applications), in this thesis we restrict ourselves to the case of linear equations only and leave the semi-linear case for future investigation. However, semi-linear equations have been considered in the literature for specific examples of cylindrical Lévy processes, for example [14], [22], [38], [41], [51], [61] etc. Further, to deal with the semi-linear case one requires some path properties like the integrability of the solution of the linear equation (see e.g. [14], [38]) and for general cylindrical Lévy processes this may be difficult. This is due to the fact that it has been observed in many examples that the solution may exist as a V-valued process but that its trajectories are highly irregular; see for example Brzeźniak et al [12], Brzeźniak and Zabczyk [14] and Peszat and Zabczyk [38]. Even in the case of genuine Lévy processes one needs some mild conditions on the generator of the semigroup to guarantee that the trajectories are càdlàg, which is the best regularity one can hope for noises with jumps. The only positive results that we are aware of on time-regularity of the paths are in Liu and Zhai [35] and Peszat and Zabczyk [39]. But these results are very restrictive and do not cover most of the considered examples of cylindrical Lévy processes. Characterising conditions which guarantee the existence of càdlàg trajectories of the solution for any cylindrical Lévy process is still an open question.

The mild solution of equation (1.0.1) can be given by the following variation of constants formula:

$$Y(t) = T(t)y_0 + \int_0^t T(t-s)B \,\mathrm{d}L(s). \tag{1.0.3}$$

This process is called the Ornstein-Uhlenbeck process driven by a cylindrical Lévy process. For the process to be V-valued, the integral on the right, a stochastic convolution integral, must exist as a V-valued random variable. The existence of such an integral requires a theory of stochastic integration of deterministic operator-valued functions with respect to the cylindrical Lévy process. In other words, we need to verify that the function $s \to T(t-s)B$ is stochastically integrable with respect to the cylindrical Lévy process. This theory of stochastic integration is developed in [46]. The process Y given by (1.0.3) will be called the mild solution of (1.0.1) and we show it also satisfies the equation (1.0.1) in the weak sense, that is, it satisfies the following weak form of (1.0.1)

$$\langle Y(t), v\rangle = \langle y_0, v\rangle + \int_0^t \langle Y(s), A^* v\rangle \, \mathrm{d}s + L(t)(B^* v), \tag{1.0.4}$$

for all v in the domain of A^*. A process Y satisfying (1.0.4) will be called weak solution of (1.0.1)

A good part of this thesis is to prove the equivalence of the mild solution and weak solution of (1.0.1). A crucial step in proving that the mild solution satisfies equation (1.0.4) is to interchange the order of a Lebesgue integral and a stochastic integral with respect to a cylindrical Lévy process. For that purpose, we need to prove a suitable version of the stochastic Fubini Theorem. It must be mentioned that an alternative approach is to employ an integration by parts formula as it is done in [16]. But proving such a formula indicates some regularity of the trajectories of the solution and is ruled out because of the fact that these trajectories, in general, may be very irregular. Therefore, to give a general theory of weak solutions, we prove in Chapter 3 an appropriate Stochastic Fubini result. Establishing such a result is challenging due to the cylindrical nature of the noise. To be more precise, the standard techniques of proving a stochastic Fubini theorem, when the integrator is a genuine Lévy process or a semi-martingale, usually rely on the Lévy-Itô decomposition or the semi-martingale decomposition of the integrator. This approach then reduces to proving a Fubini theorem for an integrator with finite moments. But the cylindrical Lévy process does not enjoy a Lévy-Itô decomposition in the underlying space.

This led us to use the approach of van Neerven and Veraar in [57] (see also [59]), where the authors proved the stochastic Fubini Theorem using the stochastic-integration theory developed using radonifying operators (e.g. [54], [58]). Using the idea from this work and without assuming any moment condition on the integrator as well as the stochastic integral, we can write the iterated integrals as an inner product in the space of square-integrable functions by using the stochastic integration theory developed in [46]. As a result of the stochastic Fubini theorem, we establish that the mild solution is a weak solution and hence we are able to generalise the theory of weak and mild solutions of the stochastic Cauchy problem driven by cylindrical Wiener processes or genuine Lévy processes to that of arbitrary cylindrical Lévy processes.

Let us consider the novelty of our method in comparison to the previous methods considered in the literature to study equation (1.0.1). So far, the stochastic Cauchy problem for cylindrical Lévy processes is studied typically using two approaches. In the first approach, the cylindrical Lévy process is given by a series of independent one-dimensional Lévy processes acting along an eigenbasis of the generator A (see (1.1.2)). A typical example considered often is the cylindrical α-stable noise. In such cases the question of existence of a weak solution reduces to the study of a sequence of one-dimensional Ornstein-Uhlenbeck processes. This approach is applied, for example, in [34], [40], [41] etc. Since this approach relies on the specific form the cylindrical Lévy process, it excludes many interesting examples of the cylindrical Lévy process. In the second approach, the cylindrical Lévy process is embedded in a larger space where it becomes a genuine Lévy process. The question then reduces to solving the equation (1.0.1) in the larger space and finding conditions for the solution to lie in the underlying space. Since these conditions may be in terms of the larger space which per se is not related to the equation under consideration, this approach also does not give a good picture of the interplay between the noise and the solution. Since our approach neither requires the cylindrical Lévy process to have a specific form nor to use other large space, it seems more natural and can be considered a direct generalisation of the genuine Lévy case or the cylindrical Wiener case.

Our approach enables us to study further properties of the solution along with some path properties. An immediate consequence of our stochastic Fubini result and its application to weak solution is that we are able to deduce that the trajectories of the solution are scalarly square integrable. We believe that this is the first positive result on an analytical path property of the solution of the stochastic Cauchy problem which is independent of the driving cylindrical Lévy process. Furthermore, without any assumptions on the cylindrical Lévy process we are able to prove that the solution process satisfies the Markov property and is stochastically continuous. In some specific examples of cylindrical Lévy processes, these properties were already proved (e.g. see [14] and [41]), but the arguments used are restricted to the specific examples under consideration. We are also able to give a condition which implies the non-existence of a modification of the solution with weakly càdlàg trajectories.

In the last part of this thesis we consider the problem of existence of an invariant measure for the solution process of (1.0.1). The existence of an invariant measure will be useful to study further properties of the solution, like the ergodicity and the second quantisation of the transition semigroup. We generalise the results of Chojnowska-Michalik in [16], [17] to the case of cylindrical Lévy processes. As far as we are aware, only for specific examples of cylindrical Lévy processes given by a series like (1.1.2), the existence of invariant measure has been considered (e.g. see [40], [41], [60]). There typically one finds the invariant measure for each of the corresponding one-dimensional Ornstein-Uhlenbeck processes and then the invariant measure for the process of (1.0.1) is obtained as the product measure of these one-dimensional invariant measures. Using the well-known log moment condition on the Lévy measures of one-dimensional Ornstein-Uhlenbeck processes and some mild assumptions on the eigenvalues of the generator of the semigroup, conditions can be obtained which guarantee existence of invariant measure. This approach is clearly very restrictive and cannot be used for all cylindrical Lévy processes.

In our general framework, having in hand the integration theory developed in [46], we are able to follow the similar approach, with some generalisations, as used for genuine Lévy processes. Similar to the genuine Lévy case, if the distributions of the stochastic convolution integral $\int_0^t T(t-s)B\,\mathrm{d}L(s)$ (which is equal in distribution to the integral $\int_0^t T(s)B\,\mathrm{d}L(s)$) converge weakly to some limit as $t \to \infty$, then the limiting distribution is an invariant measure. Therefore, the problem of existence of an invariant measure is related to the existence of the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$. Even though the noise L is cylindrical, the integral $\int_0^t T(s)B\,\mathrm{d}L(s)$ is a genuine V-valued infinitely divisible random variable by the stochastic integration theory developed in [46]. Therefore, by the compactness criterion of infinitely divisible probability measures, we can obtain the conditions for the existence of the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$. Although this looks natural and straightforward, writing conditions in terms of the cylindrical characteristics of the underlying cylindrical Lévy process is not so obvious. We need arguments to extend the cylindrical Lévy measure of the limiting distribution to a genuine Lévy measure. Such general conditions may not be easy to verify in practice but we are also able to generalise the simple log-moment condition on the Lévy measure, often considered in the literature, to a similar condition on the cylindrical Lévy measure of the cylindrical Lévy process, which is satisfied by all the examples considered in the literature referred to above.

1.1 Literature review

In this section, we give a brief review of some of the significant work done in the literature, to understand the context and importance of our work in this thesis. This review is not claimed to be exhaustive or include all related work. We concentrate on the results related to our thesis and highlight below some

(i) The case when L is a cylindrical Wiener process with covariance operator Q is well-known, for example see the monograph by Da Prato and Zabczyk [20]. In this case, the stochastic Cauchy problem (1.0.1) has a unique weak solution given by (1.0.3) if and only if

$$\int_0^T \mathrm{tr}(T(s)BQB^*T^*(s))\,\mathrm{d}s < \infty. \tag{1.1.1}$$

The solution is a mean-square continuous Markov process and the time and spatial regularity of the solution are well-understood. It is further seen (e.g. see [18]) that the solution admits an invariant measure if and only if

$$\int_0^\infty \mathrm{tr}(T(s)BQB^*T^*(s))\,\mathrm{d}s < \infty.$$

(ii) The case when L is a genuine Lévy process was first considered by Chojnowska-Michalik in [16]. In this very interesting work, the author defines a stochastic integral with respect to the Lévy process using compactness arguments and convergence in probability for square-integrable strongly measurable functions. The equivalence of the mild and weak solutions is established using an integration by parts result, and necessary and sufficient conditions for the existence of an invariant measure are derived. In particular, if the semigroup is stable it is shown that the invariant measure, if it exists, is unique. In the special case of an exponentially stable semigroup a simple sufficient (but not necessary) condition guaranteeing the existence of a stationary measure is established. In another paper [17] by the same author, necessary and sufficient simple conditions are obtained for the case of heat equation.

For the general theory of SPDEs driven by genuine Lévy processes (as well as some examples of cylindrical Lévy processes) in Hilbert spaces, one can refer to the monograph [38].

(iii) Furhman and Röckner (2000) in [24] study SCP (1.0.1) for Lévy processes in the framework of generalised Mehler semigroup and Lescot and Röckner (2004) [32] use perturbation theory to study the stochastic heat equation with $A = \Delta$ and L composed of a standard Wiener noise and an α-stable Lévy noise.

(iv) Applebaum (2006) in [2] using the martingale valued-measure and the Lévy-Itô decomposition develops a theory of stochastic integration and establish the equivalence of mild and weak solution of SCP (1.0.1) when L is a genuine Lévy process. The author also studies the relation between the concept of operator self-decomposability and the stationarity of the solution of (1.0.1).

(v) In 2010, Brzeźniak and Zabczyk [14] consider SCP (1.0.1) with $L(t) = W(\ell(t))$ and $B = \mathrm{Id}$, where W is a cylindrical Wiener process and ℓ is a real-valued Lévy subordinator. They consider the mild solution given by (1.0.3) and show that the solution can have locally unbounded trajectories.

(vi) Priola and Zabczyk (2011) in [41] consider SCP (1.0.1) with $B = \mathrm{Id}$ and

$$L(t) = \sum_{n=1}^{\infty} \beta_n l_n(t) e_n, \quad t \geqslant 0, \tag{1.1.2}$$

under the assumptions that l_n are real-valued independent, symmetric α-stable processes with $\alpha \in (0,2)$, where $(e_n) \subset D(A)$ is an orthonormal basis of V and $Ae_n = -\lambda_n e_n$, $\lambda_n > 0$, and $\lambda_n \to \infty$. By considering, for each $n \in \mathbb{N}$, the one-dimensional equation

$$\mathrm{d}Y^n(t) = -\lambda_n Y^n(t)\,\mathrm{d}t + \beta_n\,\mathrm{d}l_n(t), \quad Y^n(0) = y_n,$$

the authors prove that $Y(t) = \sum_{n=1}^{\infty} Y^n(t) e_n$ is a mild solution of SCP (1.0.1) and $Y(t) \in V$ if and only if

$$\sum_{n=1}^{\infty} \frac{\beta_n^{\alpha}}{\lambda_n} < \infty.$$

It is further shown in this specific framework that the solution is a Markov process, is stochastically continuous and P-a.s., it has trajectories in $L^p([0,T];V)$, for $0 < p < \alpha$.

(vii) Brzeźniak et al (2010) [12] consider the time regularity of the mild solution of the stochastic heat equation (1.0.1) (i.e., $A = \Delta$), where the noise L is given by (1.1.2). The authors ask when the solution takes values in V and whether the solution has càdlàg trajectories. One of their main results is that if β_n does not converge to 0, $e_n \in D(A^*)$ and Y is V-valued, then Y has no V-càdlàg modification.

(viii) Liu and Zhai in [34] answer the questions raised in [12] and [41] and in particular show that the solution of SCP (1.0.1) driven by cylindrical α-stable noise of the form (1.1.2) is càdlàg if and only if the process L is a genuine Lévy process (i.e., if and only if $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty$).

(ix) Priola and Zabczyk [40] extend the results obtained in [41] for cylindrical α-stable noise to more general cylindrical Lévy noise L given by (1.1.2) by taking l_n to be one-dimensional pure-jump symmetric Lévy process. They obtain necessary and sufficient conditions for the solution to be V-valued and to have càdlàg paths. Using some simple condition on the Lévy measure and on the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$, they obtain the existence and uniqueness of the invariant measure for the solution.

(x) Peszat and Zabczyk in [39] consider SCP (1.0.1) with $B = \mathrm{Id}$ and assuming that L is a Lévy process taking values in a larger Hilbert space $H \supset U$ and is given by

$$L(t) = \int_0^t \int_{\|u\|_H \leqslant 1} u\tilde{\pi}(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_{\|u\|_H > 1} u\pi(\mathrm{d}s, \mathrm{d}u), \tag{1.1.3}$$

where π is the Poisson random measure associated with L, and the Lévy measure ν satisfies $\nu(H \setminus U) = 0$. In Proposition 2.6, the authors give the necessary and sufficient conditions for the solution to be U-valued and to be mean-square continuous. They focus on studying the time-regularity of solutions which are further studied by Liu and Zhai (2016) in [35].

(xi) In [6], Applebaum and Riedle (2010) carry out a systematic study of cylindrical Lévy processes by generalising the cylindrical Wiener process in a natural manner. The cylindrical Lévy process is defined as a cylindrical process such that its finite dimensional projections are genuine Lévy processes. They study SCP (1.0.1) with L as a cylindrical process with weak second moments and show that there exists a unique weak cylindrical solution of (1.0.1). The solution is obtained only as a cylindrical process and not as a V-valued genuine process.

(xii) Riedle (2015) in [46] develops a theory of stochastic integration of deterministic functions and defines the Ornstein-Uhlenbeck processes driven by cylindrical Lévy processes. For stochastic integration of random integrands, one can refer to Riedle (2014) [45] for cylindrical Lévy processes with second moments and Jakubowski and Riedle (2017) [27] for general cylindrical Lévy processes.

(xiii) In [42], Riedle (2018) studies SCP (1.0.1) when L is a canonical α-stable process ($0 < \alpha \leqslant 2$), i.e. the characteristic function of $L(t)$ is given by $\varphi_{L(t)}(u) = \exp\left(-t\|u\|^{\alpha}\right)$ for all $u \in U$, and shows that (1.0.1) admits a mild solution if and only if

$$\int_0^T \|T(s)\|_{\mathrm{HS}}^{\alpha} \,\mathrm{d}s < \infty.$$

It must be mentioned here that both the definition of cylindrical Lévy process in [6] and the stochastic integration theory in [46], are introduced for Banach spaces. However, in this thesis we only consider (1.0.1) in Hilbert spaces and leave the case of Banach spaces for future research.

All of the results mentioned above on weak or mild solution, path property, Markov property, invariant measures are covered by our general theory developed in this thesis.

1.2 Brief outline of the thesis

In Chapter 2 we collect the background material, set up our notation and review some basic results from the available literature. We start by recalling the concept of trace-class and Hilbert-Schmidt operators in a Hilbert space, define C_0-semigroups and summarize some basic facts about C_0-semigroups. In Section 2.2, we give a brief summary of cylindrical measures and cylindrical random variables and we continue in Section 2.3 with the definition and examples of cylindrical Lévy processes which are central to the theme of this thesis. The next section gives a brief exposition of the integration theory developed in [46] for deterministic functions. The integration is defined in two steps, first for a regulated Hilbert space valued function and then for an operator-valued function which is weakly in the space of regulated functions. For that purpose, some basic properties of a regulated function are also recalled. Finally, the necessary and sufficient conditions for stochastic integrability of a function are stated from [46].

In Chapter 3 we prove a version of a stochastic Fubini Theorem, which plays an important role in establishing the existence of a weak solution in Chapter 4. The main result is stated in Theorem 3.2.1. For a fixed finite measure space $(S, \mathcal{S}, \eta)$ the main conclusion is to have

$$\int_S \int_0^T g(s,t)\,\mathrm{d}L(t)\eta(\mathrm{d}s) = \int_0^T \int_S g(s,t)\eta(\mathrm{d}s)\,\mathrm{d}L(t), \tag{1.2.1}$$

for a U-valued function g satisfying conditions so that both the stochastic integrals are defined. The main idea of the proof is to write the iterated integral as an inner product in the space $L^2(S) := L^2_\eta(S;\mathbb{R})$ of square integrable functions. For this purpose, we define an operator Φ which defines a regulated function from $[0,T]$ to the space of $L^2(S)$-valued Hilbert-Schmidt operators. The fact that such functions are always stochastically integrable is proved in Lemma 3.2.2. The stochastic integral $\int_0^T \Phi(t)\,\mathrm{d}L(t)$ defines an $L^2(S)$-valued random variable.

Such an integral can always be approximated by a sequence of integrals of elementary functions as is seen in Lemma 3.2.4. Using this approximation, we get the first equality below by following an idea of van Neerven and Veraar [57] and the second equality follows by the stochastic integration theory.

$$\int_S \int_0^T g(s,t)\,\mathrm{d}L(t)\eta(\mathrm{d}s) = \left\langle \int_0^T \Phi(s)\,\mathrm{d}L(s), 1 \right\rangle_{L^2(S)} = \int_0^T \int_S g(s,t)\eta(\mathrm{d}s)\,\mathrm{d}L(t).$$

In Chapter 4 we prove the existence and uniqueness of the weak solution of the stochastic Cauchy problem. In Section 4.2 we prove the stochastic continuity of the stochastic convolution process. In Theorem 4.3.1 we show that the weak solution exists if the map $s \to T(s)B$ is stochastically integrable with respect to L in $[0,T]$. In this case, the weak solution is given by the mild solution. The converse result that if the weak solution exists, then the map $s \to T(s)B$ is stochastically integrable with respect to L is proved in Theorem 4.4.2, for which we need an integration by parts formula which is proved in Lemma 4.4.1. This result also implies the uniqueness of the weak solution.

In Chapter 5, we discuss some properties satisfied by the solution process. We begin by noting in Theorem 5.0.1 that the solution process has weakly square integrable trajectories. This result is actually an immediate consequence of the stochastic Fubini theorem proved in Theorem 3.2.1. We then show in Theorem 5.0.2 that the weak solution is stochastically continuous. It was mentioned earlier that the weak solution may not have càdlàg trajectories. In Theorem 5.0.3 we give a condition on the cylindrical Lévy measure which guarantees that the weak solution does not have càdlàg trajectories. Assuming that L has weak second moments and that the weak solution has second moments, we show in Theorem 5.0.8 that the weak solution is mean-square continuous. The final result in this Chapter is Theorem 5.0.9 where we show that the weak solution is a Markov process.

Chapter 6 deals with the existence of a stationary solution of (1.0.1). In Lemma 6.2.4 it is shown that an invariant measure for the weak solution can be obtained as the distribution of the random variable $\int_0^\infty T(s)B\,\mathrm{d}L(s) := \lim_{t\to\infty}\int_0^t T(s)B\,\mathrm{d}L(s)$, if it exists, where the limit is in probability. In Theorem 6.2.8 we prove necessary and sufficient conditions for the existence of the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$. This result can be considered as a generalisation of the corresponding well-known result for genuine Lévy processes (see [16]). In Section 6.3, we consider the case of a stable semigroup. In this case, the existence of an invariant measure is equivalent to the existence of the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ (see Theorem 6.3.1) and the unique invariant measure is given by the distribution of the random variable $\int_0^\infty T(s)B\,\mathrm{d}L(s)$. If the cylindrical Lévy process L is symmetric and the semigroup is the heat semigroup, then by Theorem 6.3.4, the following conditions are necessary and sufficient for the existence of an invariant measure:

(i) $$\sup_{n\geqslant 1}\int_U \max_{1\leqslant k\leqslant n}\left(\frac{\log^+|\langle u,e_k\rangle|}{\lambda_k}\right)\mu(\mathrm{d}u)<\infty; \tag{1.2.2}$$

(ii) $$\limsup_{m\to\infty}\sup_{n\geqslant m}\int_U \max_{m\leqslant k\leqslant n}\left(\frac{\log^+|\langle u,e_k\rangle|}{\lambda_k}\right)\mu(\mathrm{d}u)=0 \tag{1.2.3}$$

These conditions can be seen as the generalisation of the corresponding condition in the case of a genuine Lévy process (see [17]). It is observed that these conditions are satisfied by the examples considered in the literature.

Chapter 2

Notation and preliminaries

In this chapter we review basic definitions and background material needed for the rest of this thesis. All the chapter is a collection of known results and definitions available in the literature.

2.1 Linear operators on Hilbert spaces and C_0-semigroups

Let U and V be real separable Hilbert spaces with norms $\|\cdot\|$ and orthonormal bases $(e_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$, respectively. We identify the dual of a Hilbert space by the space itself. The Borel σ-algebra of U is denoted by $\mathfrak{B}(U)$. By B_U, we denote the open unit ball in U, that is, $B_U := \{u \in U : \|u\| < 1\}$. The space of Radon probability measures on $\mathfrak{B}(U)$ is denoted by $\mathcal{M}(U)$ and is equipped with the Prokhorov metric. The space of all continuous functions from $[0,T]$ to U is denoted by $C([0,T];U)$ and it is equipped with the supremum norm $\|\cdot\|_\infty$. The space of all equivalence classes of measurable functions $f\colon \Omega \to U$ on a probability space $(\Omega, \mathcal{F}, P)$ is denoted by $L^0_P(\Omega;U)$, and it is equipped with the topology of convergence in probability. It is a complete metric space under the metric p given by

$$p(X,Y) = E\left[\|X-Y\| \wedge 1\right].$$

Convergence in this metric is equivalent to convergence in probability. The space $L^p_P(\Omega;U)$ contains all equivalence classes of measurable functions $f\colon \Omega \to U$ which are p-th integrable, and it is equipped with the usual norm. The space of all linear, bounded operators from U to V is denoted by $\mathcal{L}(U,V)$, equipped with the operator norm $\|\cdot\|_{\mathrm{op}}$. An operator $A \in \mathcal{L}(U,V)$ is called *nuclear* or *trace class* if there exist two sequences $(a_j)_{j\in\mathbb{N}} \subset U$ and $(b_j)_{j\in\mathbb{N}} \subset V$ such that

$$\sum_{j=1}^{\infty} \|a_j\|\|b_j\| < \infty$$

and A has the representation

$$Au = \sum_{j=1}^{\infty} \langle a_j, u\rangle b_j, \quad u \in U.$$

The space of all nuclear operators endowed with the norm

$$\|A\|_1 = \inf\left\{\sum_{j=1}^{\infty} \|a_j\|\|b_j\| : Au = \sum_{j=1}^{\infty} \langle a_j, u\rangle b_j\right\}$$

is a Banach space and is denoted by $\mathcal{L}_1(U,V)$. For any $A \in \mathcal{L}_1(U,U)$, we define

$$\mathrm{tr}A = \sum_{j=1}^{\infty} \langle Ae_k, e_k\rangle.$$

Then $\mathrm{tr}A$ is a well defined number independent of the choice of the orthonormal basis $(e_k)_{k\in\mathbb{N}}$. It can be easily seen that a non negative operator $A \in \mathcal{L}(U,U)$ is trace class if and only if for an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ in U it satisfies

$$\sum_{j=1}^{\infty}\langle Ae_k, e_k\rangle < \infty.$$

An operator $A\colon U \to V$ is called *Hilbert-Schmidt* if

$$\|A\|_{\mathrm{HS}}^2 := \sum_{k=1}^{\infty}\|Ae_k\|^2 < \infty.$$

The number $\|A\|_{\mathrm{HS}}$ is independent of the choice of the basis (e_k). Moreover $\|A\|_{\mathrm{HS}} = \|A^*\|_{\mathrm{HS}}$, where A^* denotes the adjoint operator of A. The space of all Hilbert-Schmidt operators equipped with the norm $\|\cdot\|_{\mathrm{HS}}$ is a separable Hilbert space and is denoted by $\mathcal{L}_2(U,V)$. A standard characterisation of compact sets in a Hilbert space H with orthonormal basis $(f_k)_{k\in\mathbb{N}}$ is that a set $K \subseteq H$ is compact if and only if it is closed, bounded and satisfies

$$\lim_{N\to\infty}\sup_{h\in K}\sum_{k=N+1}^{\infty}\langle h, f_k\rangle^2 = 0. \tag{2.1.1}$$

Using this characterisation of compact sets we obtain the following result.

Theorem 2.1.1. *A set $K \subseteq \mathcal{L}_2(U,V)$ is compact if and only if it is closed, bounded and satisfies*

$$\lim_{N\to\infty}\sup_{\varphi\in K}\sum_{i=N+1}^{\infty}\|\varphi e_i\|^2 = 0, \tag{2.1.2}$$

$$\lim_{N\to\infty}\sup_{\varphi\in K}\sum_{j=N+1}^{\infty}\|\varphi^* h_j\|^2 = 0. \tag{2.1.3}$$

Proof. First suppose that K is compact. Then K is closed and bounded. Suppose (2.1.2) is not true. Then there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $\varphi_N \in K$ satisfying

$$\sum_{k=N+1}^{\infty} \|\varphi_N e_k\|^2 > \varepsilon. \tag{2.1.4}$$

Since K is compact, we can find a subsequence (again denoted by $(\varphi_N)_{N\in\mathbb{N}}$) and $\varphi \in K$ such that $\varphi_N \to \varphi$ in $\mathcal{L}_2(U,V)$. Thus there exists $N_0 \in \mathbb{N}$ such that for all $N \geqslant N_0$,

$$\sum_{k=N+1}^{\infty} \|(\varphi - \varphi_N)e_k\|^2 \leqslant \|\varphi - \varphi_N\|^2_{\text{H.S.}} \leqslant \frac{\varepsilon}{2}. \tag{2.1.5}$$

Let P_N^c denotes the projection onto the span of $\{e_{N+1}, e_{N+2}, \ldots\}$. Then by (2.1.4) and (2.1.5), we obtain for all $N \geqslant N_0$

$$\begin{aligned}
\sum_{k=N+1}^{\infty} \|\varphi e_k\|^2 &= \|\varphi P_N^c\|^2_{\text{H.S.}} \\
&\geqslant |\|\varphi_N P_N^c\| - \|\varphi P_N^c - \varphi_N P_N^c\||^2 \\
&= \left| \left(\sum_{k=N+1}^{\infty} \|\varphi_N e_k\|^2 \right)^{1/2} - \left(\sum_{k=N+1}^{\infty} \|(\varphi - \varphi_N)e_k\|^2 \right)^{1/2} \right|^2 \\
&\geqslant \left(\frac{\varepsilon}{2}\right)^2.
\end{aligned}$$

This contradicts the fact that $\varphi \in \mathcal{L}_2(U,V)$. Thus (2.1.2) is satisfied. Since the map $\varphi \to \varphi^*$ is continuous, it follows that the set $\{\varphi^* : \varphi \in K\}$ is compact in $\mathcal{L}_2(V,U)$. Hence, (2.1.3) follows by the same arguments as above.

Conversely suppose that K is closed, bounded and satisfies (2.1.2) and (2.1.3). If for all $u \in U$ and $v \in V$ we define $u \otimes v \colon U \to V$ by $(u \otimes v)(h) = \langle u, h \rangle\, v$, then $(e_i \otimes h_j)_{i,j\in\mathbb{N}}$ forms an orthonormal basis of $\mathcal{L}_2(U,V)$. By the characterization (2.1.1) of compact sets in Hilbert spaces it is enough to show that

$$\lim_{N\to\infty} \sup_{\varphi\in K} \sum_{i\geqslant N+1 \text{ or } j\geqslant N+1} \langle \varphi, e_i \otimes h_j\rangle^2 = 0. \tag{2.1.6}$$

We have

$$\begin{aligned}\sup_{\varphi\in K} \sum_{i\geqslant N+1 \text{ or } j=N+1} \langle \varphi, e_i \otimes h_j\rangle^2 &\leqslant \sup_{\varphi\in K} \sum_{i=N+1}^{\infty} \sum_{j=1}^{\infty} \langle \varphi e_i, h_j\rangle^2 + \sup_{\varphi\in K} \sum_{i=1}^{\infty} \sum_{j=N+1}^{\infty} \langle \varphi e_i, h_j\rangle^2 \\ &= \sup_{\varphi\in K} \sum_{i=N+1}^{\infty} \|\varphi e_i\|^2 + \sup_{\varphi\in K} \sum_{j=N+1}^{\infty} \|\varphi^* h_j\|^2\end{aligned}$$

By using (2.1.2) and (2.1.3), we obtain (2.1.6). □

Definition 2.1.2. A family $(T(t))_{t\geqslant 0}$ of linear operators on a separable Hilbert space V is called a C_0-*semigroup* if the following conditions are satisfied:

(i) $T(0) = \mathrm{Id}$, where Id denotes the identity operator on V;

(ii) $T(t+s) = T(t)T(s)$ for all $s, t \geqslant 0$;

(iii) For every $v \in V$, $\lim_{t\to 0^+} \|T(t)v - v\| = 0$.

The *infinitesimal generator* A of a semigroup $(T(t))_{t\geqslant 0}$ with domain $\mathcal{D}(A)$ is defined as follows:

$$\begin{aligned}\mathcal{D}(A) &= \left\{ v \in V : \lim_{t\to 0^+} \frac{T(t)v - v}{t} \text{ exists} \right\} \\ Av &= \lim_{t\to 0^+} \frac{T(t)v - v}{t}, \text{ for all } v \in D(A).\end{aligned}$$

We summarise basic properties of a C_0-semigroup and its generator in the following proposition, whose proof can be found among others in [37].

Proposition 2.1.3. Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup and A be its generator. Then we have the following:

(i) There exist constants $M \geqslant 1$ and $w \in \mathbb{R}$ such that

$$\|T(t)\| \leqslant Me^{wt} \qquad \text{for all t} \geqslant 0. \tag{2.1.7}$$

(ii) For all $v \in V$, the map $t \to T(t)v$ is continuous for $t \geqslant 0$.

(iii) For all $v \in D(A)$ and $t \geqslant 0$ we have $T(t)v \in D(A)$, the map $t \to T(t)v$ is continuously differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)v = AT(t)v = T(t)Av, \qquad t \geqslant 0.$$

(iv) For all $v \in V$ we have $\int_0^t T(s)v\,\mathrm{d}s \in D(A)$ and

$$A\int_0^t T(s)v\,\mathrm{d}s = T(t)v - v.$$

If $v \in D(A)$, then both sides are equal to $\int_0^t T(s)Av\,\mathrm{d}s$.

(v) The generator A is a closed and densely defined operator.

(vi) $(T^*(t))_{t\geqslant 0}$ is a C_0-semigroup with generator A^*.

2.2 Cylindrical measures and cylindrical random variables

A cylindrical Lévy process will be defined using cylindrical measures and cylindrical random variables. In this section, we give a brief review of cylindrical measures and cylindrical random variables. One can refer to [53] for more details. Let Γ be a subset of U. Sets of the form

$$C(u_1, ..., u_n; B) := \{u \in U : (\langle u, u_1\rangle, ..., \langle u, u_n\rangle) \in B\},$$

for $u_1, ..., u_n \in \Gamma$ and $B \in \mathfrak{B}(\mathbb{R}^n)$ are called *cylindrical sets with respect to* Γ. Clearly, we can write

$$C(u_1, ..., u_n; B) = \pi^{-1}_{u_1,\cdots,u_n}(B),$$

where $\pi_{u_1,\cdots,u_n}$ is the linear map defined by

$$\pi_{u_1,\cdots,u_n} : U \to \mathbb{R}^n, \quad \pi_{u_1,\cdots,u_n}(u) := (\langle u, u_1 \rangle, ..., \langle u, u_n \rangle).$$

The set of all these cylindrical sets, denoted by $\mathcal{Z}(U, \Gamma)$, is an algebra of subsets of U and is a σ-algebra if Γ is finite ([6, Lemma 2.1]). The algebra $\mathcal{Z}(U, U)$ will be denoted by $\mathcal{Z}(U)$ and by separability of U, the σ-algebra generated by $\mathcal{Z}(U)$ coincides with the Borel σ-algebra $\mathfrak{B}(U)$ [53, Theorem I.2.1].

Definition 2.2.1. A function $\mu\colon \mathcal{Z}(U) \to [0, \infty]$ is called a *cylindrical measure*, if for each finite subset $\Gamma \subseteq U$ the restriction of μ on the σ-algebra $\mathcal{Z}(U, \Gamma)$ is a measure.

A cylindrical measure is only finitely additive and by definition its restriction to $\mathcal{Z}(U, \Gamma)$ for each finite subset $\Gamma \subseteq U$ is countably additive. If $B\colon U \to V$ is a mapping such that for every $C \in \mathcal{Z}(V)$, the pre-image $B^{-1}(C) \in \mathcal{Z}(U)$, then the mapping $C \to \mu(B^{-1}(C))$ defines a cylindrical measure on $\mathcal{Z}(V)$, denoted by $\mu \circ B^{-1}$, and is called the image of the cylindrical measure under B. It is well known that if μ is a cylindrical measure on $\mathcal{Z}(U)$ and B is a Hilbert-Schmidt operator from U to V, then the image measure $\mu \circ B^{-1}$ on $\mathcal{Z}(V)$ extends to a Radon measure on $\mathfrak{B}(V)$ [53, Theorem VI.5.2]. Note that a cylindrical measure μ on $\mathcal{Z}(U)$ is said to extend to a measure ν on $\mathfrak{B}(U)$ if $\mu = \nu$ on $\mathcal{Z}(U)$. Obviously, Radon extension of a cylindrical measure is countably additive and unique. Not every cylindrical measure on $\mathcal{Z}(U)$ can be extended to a Radon measure on $\mathcal{B}(U)$ (see Example 2.2.5 given below). By Caratheodory's extension theorem it is possible if and only if the cylindrical measure is countably additive on $\mathcal{Z}(U)$. An obvious consequence of the definition of the cylindrical measure is that $\mu \circ \pi_{u_1,\dots,u_n}^{-1}$ is a measure on $\mathfrak{B}(\mathbb{R}^n)$ for any $u_1, \dots, u_n \in U$. A cylindrical measure is called finite if $\mu(U) < \infty$ and a cylindrical probability measure if $\mu(U) = 1$.

For any function $f\colon U \to \mathbb{C}$ which is $\mathcal{Z}(U, \Gamma)$-measurable where Γ is a finite subset of U, one can define the integral with respect to μ as a Lebesgue integral if it exists. As a consequence, the *characteristic function* of a finite cylindrical measure μ on $\mathcal{Z}(U)$ can be defined by

$$\varphi_\mu\colon U \to \mathbb{C}, \qquad \varphi_\mu(u) = \int_U e^{i\langle u,h\rangle} \mu(\mathrm{d}h).$$

A cylindrical (probability) measure μ is called *continuous* if the characteristic function φ_μ is continuous [53]. A cylindrical measure μ is called symmetric if $\mu(-C) = \mu(C)$ for each $C \in \mathcal{Z}(U)$. The symmetry of μ is equivalent to the condition that the characteristic function φ_μ is real-valued. The convolution of two cylindrical measures μ_1 and μ_2 (see [47]), denoted by $\mu_1 * \mu_2$, is a cylindrical measure defined for any cylindrical set $C = \{u \in U : (\langle u, u_1\rangle, \dots, \langle u, u_n\rangle) \in B\}$ by

$$(\mu_1 * \mu_2)(C) = \int_U \mu_1(C - u)\mu_2(\mathrm{d}u).$$

The characteristic function satisfies $\varphi_{\mu_1 * \mu_2}(u) = \varphi_{\mu_1}(u)\varphi_{\mu_2}(u)$ for all $u \in U$. By μ^{*n} we denote the n-times convolution of μ with itself. A (cylindrical) probability measure μ is called infinitely divisible if for all $n \in \mathbb{N}$, there exists a (cylindrical) probability measure $\mu^{1/n}$ such that $\mu = \left(\mu^{1/n}\right)^{*n}$.

The following result on relative compactness will be useful later.

Proposition 2.2.2. If μ is a continuous cylindrical probability measure on $\mathcal{Z}(U)$ and K is a compact subset of $\mathcal{L}_2(U, V)$, then the set $\{\mu \circ \varphi^{-1} : \varphi \in K\}$ is relatively compact in the space $\mathcal{M}(V)$ of probability measures on $\mathfrak{B}(V)$.

Proof. See Proposition 5.3 in [27]. □

Definition 2.2.3. A *cylindrical random variable* Z in U is defined as a linear and continuous map

$$Z\colon U \to L_P^0(\Omega;\mathbb{R}).$$

The concept of a cylindrical random variable is well-established in the literature (e.g. see [8], [9], [25], [30], [50], [53]) appearing in different guises, e.g., the generalised random function or the weak distributions. It is sometimes assumed to be only linear and sometimes as a mapping into $L_P^p(\Omega;\mathbb{R})$ for some $p > 0$. Each cylindrical random variable Z defines a continuous cylindrical probability measure λ by

$$\lambda\colon \mathcal{Z}(U) \to [0,1], \qquad \lambda(C) = P\big((Zu_1,\dots,Zu_n) \in B\big)$$

for cylindrical sets $C = C(u_1, ..., u_n; B)$. The cylindrical probability measure λ is called the *cylindrical distribution* of Z. Conversely, if μ is a continuous cylindrical probability measure, then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a cylindrical random variable with cylindrical distribution μ. The characteristic function of a cylindrical random variable Z is defined by

$$\varphi_Z\colon U \to \mathbb{C}, \qquad \varphi_Z(u) = E[\exp(iZu)],$$

and it uniquely determines the cylindrical distribution of Z.

Example 2.2.4. Let $X\colon \Omega \to U$ be a (classical or genuine) U-valued random variable, that is, X is $\mathcal{F}$-$\mathfrak{B}(U)$-measurable function. Then

$$Z\colon U \to L_P^0(\Omega;\mathbb{R}), \qquad Zu := \langle X, u\rangle$$

defines a cylindrical random variable. We then say that Z is induced by X. In this case the cylindrical distribution of Z extends to a Radon probability measure. It is also clear that the characteristic function of cylindrical random variable Z coincides with the characteristic function of X. Indeed, this is also a sufficient condition for a cylindrical random variable to be induced by a classical random variable, that is, the characteristic function of a cylindrical random variable Z in U coincides with the characteristic function of a Radon probability measure if and only if it is induced by a U-valued random variable [53, Theorem IV.2.5].

In general, not every cylindrical random variable can be induced by a classical random variable.

Example 2.2.5. Let $\varphi\colon U \to \mathbb{C}$ be defined by $\varphi(u) = \exp\left(-\frac{1}{2}\|u\|^2\right)$. Then φ being symmetric, continuous and positive definite, is a characteristic function of a cylindrical probability measure γ, called the *canonical Gaussian cylindrical measure*. If U is infinite dimensional, then φ is not a characteristic function of a probability measure because it is not weakly sequentially continuous. Therefore, γ is not countably additive. The cylindrical random variable Z with cylindrical distribution γ can not be induced by a classical random variable.

A family $(Z(t) : t \geqslant 0)$ of cylindrical random variables $Z(t)$ is called a *cylindrical process*.

2.3 Cylindrical Lévy processes: definition and examples

In this section we define and give examples of a cylindrical Lévy process. But before that let us recall the definition and basic properties of a Lévy process in a Hilbert space. For details and other properties one can refer to [4] (for finite dimensional Lévy processes) and [38] for infinite dimensional Lévy processes.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a filtered probability space where the filtration satisfies the usual conditions of right continuity and completeness. In this thesis we use the following definition of a Lévy process.

Definition 2.3.1 (Lévy Process)**.** A U-valued stochastic process $(X(t) : t \geqslant 0)$, defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$, is called a *Lévy process* in U if

(a) $X(0) = 0$ P-a.s.;

(b) it has stationary and independent increments, where the independence is with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, i.e., for all $s \leqslant t$, the increment $X(t) - X(s)$ is independent of $\mathcal{F}_s$;

(c) it is stochastically continuous, i.e. for any $\varepsilon > 0$ and for all $t_0 > 0$,

$$\lim_{t\to t_0} P(|X(t) - X(t_0)| > \varepsilon) = 0.$$

By Theorem 4.3 in [38] (or Theorem 2.1.8 in [4]), for every Lévy process there exists a modification with P-a.s. càdlàg (right continuous with left limits) paths. Some standard examples of Lévy processes are Brownian motions, Poisson Processes, compound Poisson processes and α-stable processes. The characteristic function of a Lévy process is given by the Lévy-Khintchine formula

$$\varphi_{X(t)} \colon U \to \mathbb{C}, \qquad \varphi_{X(t)}(u) := E[e^{i\langle X(t),u\rangle}] = e^{t\eta(u)}, \tag{2.3.1}$$

where

$$\eta(u) = i\langle b, u\rangle - \frac{1}{2}\langle u, Qu\rangle + \int_U \left(e^{i\langle y,u\rangle} - 1 - i\langle y, u\rangle \mathbb{1}_{B_U}(y)\right) \nu(\mathrm{d}y), \tag{2.3.2}$$

where $b \in U$, $Q \colon U \to U$ is a positive-definite symmetric trace class operator, ν is a Lévy measure on U, that is, a σ-finite measure satisfying $\nu(\{0\}) = 0$ and

$$\int_U \left(\|u\|^2 \wedge 1\right) \nu(\mathrm{d}u) < \infty.$$

Every Lévy process enjoys the famous Lévy-Itô decomposition into a continuous and jump part. Let $\Delta X(t) := X(t) - X(t-)$ be the jump of X at t and define a Poisson random measure N by

$$N(t, A)(\omega) := \#\{0 \leqslant s \leqslant t; \Delta X(s)(\omega) \in A\} = \sum_{0<s\leqslant t} \mathbb{1}_A(\Delta X(s)(\omega)),$$

for $A \in \mathcal{B}(U)$ such that $0 \notin \bar{A}$.

Theorem 2.3.2. *Let X be a U-valued Lévy process. Then there exists $b \in U$, a U-valued Wiener process W, a Poisson random measure N on $[0,\infty) \times U$ with intensity measure $\mathrm{d}t \otimes \nu$, where ν is a Lévy measure, such that*

$$X(t) = bt + W(t) + \int_{\|u\|<1} u\tilde{N}(t, \mathrm{d}u) + \int_{\|u\|\geqslant 1} uN(t, \mathrm{d}u)$$

where $\tilde{N}(t, \mathrm{d}u) := N(t, \mathrm{d}u) - t\nu(\mathrm{d}u)$ is the compensated Poisson random measure.

Proof. See [4, Theorem 2.4.16] or [38, Theorem 4.23]. □

Each Lévy process is an infinitely divisible process, that is, for each t, the distribution of $L(t)$ is an infinitely divisible probability measure. Indeed, the characteristic function of any infinitely divisible probability measure is given by the Lévy-Khintchine formula given in (2.3.1) (with t=1). Conversely, corresponding to every function $\varphi \colon U \to \mathbb{C}$ of the form $\varphi(\cdot) = e^{\eta(\cdot)}$, where η is the function defined in (2.3.2), there exists an infinitely divisible probability measure θ whose characteristic function is given by φ. The triple (b, Q, ν) will also be called the characteristics of θ. The following compactness criterion for infinitely divisible probability measures in Hilbert spaces will be needed in the thesis. Its proof can be found in Theorem VI.5.3 of [36].

Theorem 2.3.3. *A family $(\lambda_\varphi)_{\varphi\in\Lambda}$ of infinitely divisible probability measures with characteristics $(a_\varphi, Q_\varphi, \nu_\varphi)$ in a Hilbert space U is relatively compact in $\mathcal{M}(U)$ if and only if the following three conditions are satisfied:*

(a) the set $\{a_\varphi : \varphi \in \Lambda\} \subset U$ *is relatively compact;*

(b) the set $\{\nu_\varphi : \varphi \in \Lambda\}$ *restricted to the complement of any neighbourhood of the origin is relatively compact;*

(c) the operators $T_\varphi : U \to U$ *defined by*

$$\langle T_\varphi v, v\rangle := \langle Q_\varphi v, v\rangle + \int_{\|h\|\leqslant 1} \langle v, h\rangle^2 \nu_\varphi(\mathrm{d}h)$$

satisfy the conditions

(i) $\sup\limits_{\varphi\in\Lambda} \sum\limits_{k=1}^{\infty} \langle T_\varphi e_k, e_k\rangle < \infty,$

(ii) $\lim\limits_{N\to\infty} \sup\limits_{\varphi\in\Lambda} \sum\limits_{k=N}^{\infty} \langle T_\varphi e_k, e_k\rangle = 0.$

We are now in a position to introduce a cylindrical Lévy process as defined by Applebaum and Riedle in [6]. A cylindrical Wiener process (an example of cylindrical Lévy process) has been known for a long time and there is a vast literature on SPDE's driven by a cylindrical Wiener process (e.g., [20]). In the literature, many noises have been considered as analogues for the cylindrical Wiener process in the discontinuous or non-Gaussian settings (see e.g. [12], [14], [34] [38], [41], [40]). A systematic study of cylindrical Lévy processes is carried out in [6], where the authors give the following definition, which can be considered as a natural generalisation of the concept of cylindrical Weiner process. This definition includes all examples of such a noise considered in the literature mentioned above.

Definition 2.3.4 (Applebaum and Riedle, 2010)**.** A cylindrical process $(L(t) : t \geqslant 0)$ is called a *cylindrical Lévy process* in U if for all $u_1, ..., u_n \in U$ and $n \in \mathbb{N}$,

$$((L(t)u_1, ..., L(t)u_n) : t \geqslant 0)$$

is a Lévy process in $\mathbb{R}^n$ with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$.

Remark 2.3.5. Note that the above definition is slightly stronger than that given in [6].

The characteristic function of $L(t)$ for all $t \geqslant 0$ is given by

$$\varphi_{L(t)} \colon U \to \mathbb{C}, \qquad \varphi_{L(t)}(u) = \exp\big(t\Psi(u)\big),$$

where $\Psi \colon U \to \mathbb{C}$ is called the symbol of L, and is of the form

$$\Psi(u) = ia(u) - \frac{1}{2}\langle Qu, u\rangle + \int_U \Big(e^{i\langle u,h\rangle} - 1 - i\langle u,h\rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle u,h\rangle)\Big)\, \mu(\mathrm{d}h),$$

where $a \colon U \to \mathbb{R}$ is a continuous mapping with $a(0) = 0$, $Q \colon U \to U$ is a positive, symmetric operator and μ is a cylindrical measure on $\mathcal{Z}(U)$ satisfying

$$\int_U \big(\langle u,h\rangle^2 \wedge 1\big)\, \mu(\mathrm{d}h) < \infty \qquad \text{for all } u \in U. \tag{2.3.3}$$

We call (a, Q, μ) the *(cylindrical) characteristics of L.*

A cylindrical measure satisfying (2.3.3) will be called a *cylindrical Lévy measure*. By Lemma 3.1 in [46], it follows that a cylindrical Lévy measure μ of a cylindrical Lévy process satisfies for any $c > 0$,

$$\sup_{\|u\|\leqslant c} \int_U \big(\langle u,h\rangle^2 \wedge 1\big)\, \mu(\mathrm{d}h) < \infty. \tag{2.3.4}$$

Further, by Lemma 4.4 in [44], for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset U$ satisfying $u_n \to u$ in U, it follows that

$$(|\beta|^2 \wedge 1)\mu \circ \langle \cdot, u_n\rangle^{-1} \to (|\beta|^2 \wedge 1)\mu \circ \langle \cdot, u\rangle^{-1} \quad \text{weakly in } \mathcal{M}(\mathbb{R}). \tag{2.3.5}$$

We now give some examples of cylindrical Lévy processes.

Example 2.3.6. (Cylindrical Wiener process). A cylindrical process $(W(t) : t \geqslant 0)$ is called a *cylindrical Wiener process* in U if for all $u_1, ..., u_n \in U$ and $n \in \mathbb{N}$,

$$((W(t)u_1, ..., W(t)u_n) : t \geqslant 0)$$

is a Wiener process in $\mathbb{R}^n$. The cylindrical Wiener process is a cylindrical Lévy process with characteristics $(0, Q, 0)$ where Q is the covariance operator of W. The cylindrical Wiener process $(W(t) : t \geqslant 0)$ is U-valued if it is induced by a U-valued Wiener process. It is well-known that in that case Q is a trace class operator. If $Q = \mathrm{Id}$, we call the cylindrical Wiener process a standard cylindrical Brownian motion. There are many definitions of cylindrical Wiener processes in literature. See Example 2.3.8 below for an alternative representation as a series of one-dimensional Wiener processes. For connections between different approaches and some other interesting properties see [43].

Example 2.3.7. (Genuine Lévy process). Let $(X(t) : t \geqslant 0)$ be a U-valued Lévy process with characteristics (b, Q, ν). Then

$$L(t)\colon U \to L^0_P(\Omega; \mathbb{R}); \qquad L(t)u := \langle X(t), u\rangle,$$

defines a cylindrical Lévy process with characteristics (a, Q, ν) where

$$a(u) = \langle b, u\rangle + \int_U \langle u, h\rangle \left(\mathbb{1}_{B_{\mathbb{R}}}(\langle u, h\rangle) - \mathbb{1}_{B_U}(h)\right) \nu(\mathrm{d}h).$$

Example 2.3.8. Let $(l_k)_{k\in\mathbb{N}}$ be a sequence of independent real-valued Lévy processes with characteristics (b_k, r_k, μ_k). If for each $t \geqslant 0$ and $u \in U$, the sum

$$L(t)u := \sum_{k=1}^{\infty} \langle e_k, u\rangle l_k(t)$$

converges P-a.s., and $\{\varphi_{l_k(1)} : k \in \mathbb{N}\}$ is equicontinuous at 0, then L defines a cylindrical Lévy process in U. By Lemma 4.2 in [46], the cylindrical characteristics (a, Q, μ) of L are given by

$$\begin{aligned} a(u) &= \sum_{k=1}^{\infty} \langle e_k, u\rangle \left(b_k + \int_{\mathbb{R}} \beta \left(\mathbb{1}_{B_{\mathbb{R}}}(\langle e_k, u\rangle \beta) - \mathbb{1}_{B_{\mathbb{R}}}(\beta)\right) \nu_k(\mathrm{d}\beta) \right), \\ Qu &= \sum_{k=1}^{\infty} \langle e_k, u\rangle r_k e_k, \\ \mu \circ \langle \cdot, u\rangle^{-1} (\mathrm{d}\beta) &= \sum_{k=1}^{\infty} \left(\mu_k \circ m_k(u)^{-1}\right)(\mathrm{d}\beta), \end{aligned}$$

for each $u \in U$, where $m_k(u)\colon \mathbb{R} \to \mathbb{R}$ is defined by $m_k(u)(\beta) := \beta \langle u, e_k\rangle$. Some special cases of this example are worth mentioning. If for each $t \geqslant 0$, $l_k(t) := \sigma_k l'_k(t)$, where l'_k are independent and identically distributed and $\sigma = (\sigma_k)_{k\in\mathbb{N}} \subset \mathbb{R}$, then L defines a cylindrical Lévy process in each of the following cases:

- l'_k is a Wiener process and $\sigma \in l^{\infty}$. ([46, Example 4.3]).
- l'_k is a Poisson process with intensity 1 and $\sigma \in l^2$. ([46, Example 4.4])
- l'_k is a symmetric standardised α-stable process and $\sigma \in l^{\frac{2\alpha}{2-\alpha}}$. ([46, Example 4.5]) This example is considered often in literature and usually called cylindrical α-stable noise and was considered first in [41].

Remark 2.3.9. In the above example, if for each $t \geqslant 0$, the series $\sum_{k=1}^{\infty} l_k(t)e_k$ converges

P-a.s. in U, then the process $(X(t) : t \geqslant 0)$ defined by

$$X(t) : \Omega \to U, \qquad X(t) = \sum_{k=1}^{\infty} l_k(t) e_k$$

is a Lévy process in U (see [38]) and we have

$$L(t)u = \langle X(t), u \rangle .$$

In this case, the cylindrical Lévy process L is induced by a classical or genuine Lévy process in U. Thus, corresponding to the cases mentioned in above example, in each of the following cases L is induced by a classical Lévy process:

- l'_k is Wiener process and $\sigma \in l^2$. ([46, Example 4.3])
- l'_k is a Poisson process with intensity 1 and $\sigma \in l^1$. ([46, Example 4.4])
- l'_k is a symmetric standardised α-stable process and $\sigma \in l^\alpha$. ([46, Example 4.5])

Example 2.3.10. Let L be the canonical α-stable cylindrical Lévy process for $\alpha \in (0, 2)$ considered in [42], i.e. the characteristic function of $L(t)$ is of the form

$$\varphi_{L(t)} : U \to \mathbb{C}, \qquad \varphi_{L(t)}(u) = \exp\left(-t\|u\|^\alpha\right).$$

Each finite dimensional projection $\big((L(t)u_1, \dots, L(t)u_n) : t \geqslant 0\big)$ for $u_1, \dots, u_n \in U$ is an α-stable cylindrical Lévy process in $\mathbb{R}^n$. Then by [42, Lemma 2.4] the characteristics of L is given by $(0, 0, \mu)$, where the cylindrical Lévy measure μ is symmetric and satisfies for all $n \in \mathbb{N}$:

$$\mu \circ \pi^{-1}_{e_1, \cdots, e_n}(B) = \frac{\alpha}{c_\alpha} \int_{S(\mathbb{R}^n)} \lambda_n(\mathrm{d}\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{1}{r^{1+\alpha}} \,\mathrm{d}r \qquad \text{for } B \in \mathfrak{B}(\mathbb{R}^n),$$

where λ_n is uniformly distributed on the sphere $S(\mathbb{R}^n) := \{\beta \in \mathbb{R}^n : |\beta| = 1\}$ with

$$\lambda_n\left(S(\mathbb{R}^n)\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)},$$

and c_α is a constant (see [42, Theorem A.1]) and Γ denotes the Gamma function.

In [42], it is shown that the canonical α-stable process enjoys two representations. One as a Lévy space-time white noise as defined in [1] or [7] and the second as a cylindrical standard Brownian motion subordinated by an α-stable noise. The second representation is a specific case of the following more general subordinated Wiener noise defined in [14]:

Example 2.3.11. Let W be a cylindrical Wiener process in U with covariance operator R, l be a real-valued Lévy subordinator (i.e. a one-dimensional non-decreasing Lévy process) with characteristics $(\alpha, 0, \rho)$, independent of W. Then, for each $t \geqslant 0$,

$$L(t)u := W(l(t))u,$$

defines a cylindrical Lévy process $(L(t) : t \geqslant 0)$ in U. By Lemma 4.8 in [46], the characteristics of L are given by $(0, Q, \mu)$ where

$$Q = \alpha R, \qquad \mu = (\gamma \otimes \rho) \circ \kappa^{-1},$$

where γ is the canonical Gaussian cylindrical measure on the reproducing kernel Hilbert space H_R of R with embedding $i_R\colon H_R \to U$ and

$$\kappa\colon H_R \times \mathbb{R}_+ \to U, \quad \kappa(h, s) := \sqrt{s}\, i_R h.$$

2.4 Integration with respect to cylindrical Lévy processes

We now review the theory of stochastic integration of a deterministic function $f\colon [0,T] \to \mathcal{L}(U,V)$, with respect to a cylindrical Levy process $(L(t) : t \geqslant 0)$ in U with characteristics (a,Q,μ), as developed by Riedle in [46]. Let $R([0,T],U)$ denote the space of deterministic regulated functions. Here, a function $g\colon [0,T] \to U$ is called regulated if for every $t \in (0,T)$, both the left and right limits at t, namely $g(t-)$ and $g(t+)$, exist along with the one-sided limits $g(0+)$ and $g(T-)$. Some properties of a regulated function which we will use frequently in this thesis are given in the following theorem (See [11, Ch.II.1.3], Problem 1 in [21, Ch.VII.6] and [21, 7.6.1]):

Theorem 2.4.1. *If $g\colon [0,T] \to U$ is regulated, then*

(a) g has only discontinuities of the first kind. In particular, g has only a countable number of discontinuities.

(b) The set $\overline{\{g(t) : t \in [0,T]\}}$ is a compact set in U, where $\overline{A}$ denotes the closure of subset A in U.

(c) There exists a sequence $\{(t_k^n)_{k=0}^{N_n} : n \in \mathbb{N}\}$ of partitions of $[0,T]$ such that $\max\limits_{0\leqslant k\leqslant N_n-1} |t_{k+1}^n - t_k^n| \to 0$ and that the functions

$$g_n(t) := \begin{cases} g\left(\frac{t_k^n + t_{k+1}^n}{2}\right), & \text{if } t \in (t_k^n, t_{k+1}^n),\ k = 0,\dots,N_n-1, \\ g(t_k^n), & \text{if } t = t_k^n,\ k = 0,\dots,N_n, \end{cases}$$

satisfy

$$\sup_{t\in[0,T]} \|g(t) - g_n(t)\| \to 0 \qquad \text{as } n \to \infty.$$

By [11, Ch.II.1.3], a necessary and sufficient condition for a function g to be regulated is that it can be uniformly approximated by step functions. The space $R\left([0,T],U\right)$ when equipped with the supremum norm results in a Banach space and the space $S([0,T];U)$ of step functions is dense in $R\left([0,T],U\right)$.

To define the stochastic integral of an operator-valued function with respect to a cylindrical Lévy process, first an integral for a U-valued regulated function is introduced in the following way.

Let $g \in S([0,T];U)$ be a step function. Then there exists a partition $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_m = T$ such that g takes some constant value u_k on the open interval (t_k, t_{k+1}) for $k = 0, \ldots, m-1$. Define a mapping $J\colon S([0,T];U) \to L_P^0(\Omega;\mathbb{R})$ by

$$J(g) := \sum_{k=0}^{m-1} \left(L(t_{k+1}) - L(t_k)\right)(u_k).$$

The operator $J\colon S([0,T];U) \to L_P^0(\Omega;\mathbb{R})$ is continuous and hence can be extended to the space $R\left([0,T],U\right)$, by defining

$$J(g) := \lim_{n\to\infty} J(g_n) \qquad \text{in } L_P^0(\Omega;\mathbb{R}),$$

where $(g_n)_{n\in\mathbb{N}} \subset S([0,T];U)$ is chosen such that $g_n \to g$ in $R([0,T];U)$. We will denote $J(g)$ by $\int_0^T g(s)\,\mathrm{d}L(s)$. The characteristic function of the random variable $\int_0^T g(s)\,\mathrm{d}L(s)$ is given by

$$\varphi_{J(g)}\colon \mathbb{R} \to \mathbb{C}, \qquad \varphi_{J(g)}(\beta) = \exp\left(\int_0^T \Psi(\beta g(s))\,\mathrm{d}s\right),$$

where Ψ is the cylindrical Lévy symbol of L. Using the convergence of the characteristic functions, we obtain the following result (see Lemma 5.2 in [46]):

Lemma 2.4.2. If g_n converges to g in $R([0,T];U)$, then

$$\lim_{n\to\infty} \int_0^T g_n(s)\,\mathrm{d}L(t) = \int_0^T g(s)\,\mathrm{d}L(t) \qquad \text{in probability.}$$

Proof. For each $\beta \in \mathbb{R}$ we have

$$\begin{aligned} E\left[\exp\left(i\beta \int_0^T \big(g_n(s) - g(s)\big)\,\mathrm{d}L(t)\right)\right] \\ = \exp\left(\int_0^T \Psi\big(\beta\,(g_n(s) - g(s))\,\big)\,\mathrm{d}t\right), \end{aligned} \tag{2.4.1}$$

where Ψ denotes the Lévy symbol of L. Since g_n converges to g in $R([0,T];U)$

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|g_n(s) - g(s)\| < \infty.$$

By Lemma 3.2 in [46] Ψ is continuous and maps bounded sets to bounded sets, therefore it follows by Lebesgue's theorem on dominated convergence that

$$\lim_{n\to\infty} \int_0^T \Psi\left(\beta(g_n(s) - g(s))\right)\,\mathrm{d}t = 0.$$

Consequently by (2.4.1) it follows that $\int_0^T (g_n(s) - g(s))\,\mathrm{d}L(s) \to 0$ in distribution and hence in probability. □

A function $f : [0,T] \to \mathcal{L}(U,V)$ is called weakly in $R([0,T];U)$ if $f^*(\cdot)v$ is in $R([0,T];U)$ for each $v \in V$. For such a function, one can define by using the above construction, for each $A \in \mathfrak{B}([0,T])$, a cylindrical random variable

$$Z_A\colon V \to L^0_P(\Omega;\mathbb{R}), \qquad Z_A v = \int_0^T \mathbb{1}_A(t) f^*(t) v\,\mathrm{d}L(t).$$

A function $f\colon [0,T] \to \mathcal{L}(U,V)$ is called *stochastically integrable with respect to L* if f is weakly in $R([0,T];U)$ and if for each $A \in \mathfrak{B}([0,T])$ there exists a V-valued random variable I_A such that

$$\langle I_A, v\rangle = Z_A v \qquad \text{for all } v \in V.$$

The stochastic integral I_A is also denoted by $\int_A f(s)\,\mathrm{d}L(s) := I_A$. From the very definition it follows that

$$\left\langle \int_A f(s)\,\mathrm{d}L(s), v \right\rangle = \int_A f^*(s)v\,\mathrm{d}L(s) \qquad \text{for all } v \in V. \tag{2.4.2}$$

The characteristic function of the cylindrical random variable Z_A (and hence of I_A, if it exists) is given by

$$\varphi_{Z_A}\colon V \to \mathbb{C}, \qquad \varphi_{Z_A}(v) = \exp\left(\int_A \Psi(f^*(s)v)\,\mathrm{d}s\right).$$

The existence of I_A requires that the cylindrical distribution of Z_A extends to a Radon probability measure, that is, there exists a Radon probability measure whose characteristic function coincides with the characteristic function of Z_A (see [53, Theorem IV.2.5]). Using the form of the characteristic function given above, and noting that I_A if it exists must be infinitely divisible, necessary and sufficient conditions for stochastic integrability of a function f are obtained in [46]. In particular, we have the following result.

Theorem 2.4.3. *A function $f\colon [0,T] \to \mathcal{L}(U,V)$, which is weakly in $R([0,T];U)$, is stochastically integrable with respect to a cylindrical Lévy process with characteristics (a,Q,μ) if and only if the following is satisfied:*

(1) for every sequence $(v_n)_{n\in\mathbb{N}} \subseteq V$ converging weakly to 0 and $A \in \mathfrak{B}([0,T])$ we have

$$\lim_{n\to\infty} \int_A a(f^*(s)v_n)\,\mathrm{d}s = 0. \tag{2.4.3}$$

(2) $$\int_0^T tr\big[f(t)Qf^*(t)\big]\,\mathrm{d}t < \infty; \tag{2.4.4}$$

(3) $$\limsup_{m\to\infty} \sup_{n\geqslant m} \int_0^T \int_U \left(\sum_{k=m}^n \langle f(t)u, h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)\,\mathrm{d}t = 0. \tag{2.4.5}$$

Proof. See Theorem 5.10 and Lemma 5.8 in [46]. □

Condition (2.4.3) can be replaced by the following condition (see Lemma 5.8 in [46]):

(1') the mapping T_a is weak-weakly sequentially continuous where

$$T_a \colon V \to L^1([0,T];\mathbb{R}), \quad T_a v = a(f^*(\cdot)v);$$

It is well-known (e.g. see [20]) that the stochastic integral can be defined for a deterministic operator-valued function w.r.t. a cylindrical Wiener Process W with covariance Q. The space of stochastically integrable functions in this case are the operator-valued functions f which satisfy the following:

$$\int_0^T \mathrm{tr}(f(t)Qf^*(t))\,\mathrm{d}t = \int_0^T \|f(t)\circ Q^{1/2}\|^2_{L_2(U,V)}\,\mathrm{d}t < \infty.$$

Noting that the characteristics of W are $(0,Q,0)$, the above result reduces to this already well-known result. Chojnowska-Michalik in [16] defined the stochastic integral of a strongly measurable function $f\colon [0,T] \to \mathcal{L}(U,V)$ satisfying $\int_0^T \|f(t)\|^2_{\mathrm{op}}\,\mathrm{d}t < \infty$, w.r.t. a genuine Lévy process using convergence in probability and the weak compactness arguments. It can be easily seen that such a function satisfies the necessary and sufficient conditions of Theorem 2.4.3.

Chapter 3

Stochastic Fubini Theorem

In this Chapter, we prove a stochastic version of Fubini theorem, which plays an important role in proving the existence of solution of the stochastic Cauchy problem in the next Chapter. This is based on joint work with my supervisor Prof. Markus Riedle available on arxiv and closely follows Section 3 in the preprint [31].

3.1 Introduction

In the proof of the existence of a solution, we are required to interchange the order of a stochastic integral and a Lebesgue integral. In particular, we need to establish the following equality

$$\int_0^T \int_0^T 1_{[0,s]}(r) B^* T^*(s-r) A^* v^* \, \mathrm{d}L(s) \, \mathrm{d}r = \int_0^T \int_0^T 1_{[0,s]}(r) B^* T^*(s-r) A^* v^* \, \mathrm{d}r \, \mathrm{d}L(s).$$

The integrands of the stochastic integrals on both sides of this equality are regulated functions. Therefore, it is natural for us to establish a Fubini-type result where the integrand satisfies the properties shared by the integrand in the above equality. Establishing such a result is the main aim of this Chapter. As the cylindrical Lévy process L does not enjoy a Lévy-Itô decomposition in U we cannot exploit standard arguments. We are motivated by the approach as given in [57].

In the rest of the thesis we fix a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$, where $(\mathcal{F}_t)_{t\geqslant 0}$ is a filtration satisfying the usual hypothesis. L is a cylindrical Lévy process w.r.t. $(\mathcal{F}_t)_{t\geqslant 0}$. We will always denote by (a, Q, μ) the characteristics of L.

Let $(S, \mathcal{S}, \eta)$ be a finite measure space and $L^2_\eta(S; U)$ the Bochner space.

3.2 Main result

The following version of stochastic Fubini theorem is the main result of this Chapter.

Theorem 3.2.1. *Let $g\colon S \times [0,T] \to U$ be a function satisfying the following assumptions:*

(a) *g is $\mathcal{S} \otimes \mathfrak{B}([0,T])$ measurable;*

(b) *the map $t \mapsto g(s,t)$ is regulated for η-almost all $s \in S$;*

(c) *the map $t \mapsto g(\cdot, t)$ belongs to $R([0,T]; L^2_\eta(S;U))$.*

Then, P-almost surely, we have

$$\int_S \int_0^T g(s,t)\, \mathrm{d}L(t)\, \eta(\mathrm{d}s) = \int_0^T \int_S g(s,t)\, \eta(\mathrm{d}s)\, \mathrm{d}L(t),$$

and all integrals are well defined, i.e.

(1) *the map $t \mapsto \int_S g(s,t)\, \eta(\mathrm{d}s)$ is in $R([0,T]; U)$;*

(2) *the process $\left(\int_0^T g(s,t)\, \mathrm{d}L(t) : s \in S \right)$ defines a random variable in $L^2_\eta(S; \mathbb{R})$.*

We divide the proof of the theorem into several lemmas. The theory of integration developed in [46] and recalled in Section 2.4 applies to deterministic integrands $\Phi\colon [0,T] \to \mathcal{L}(U,V)$

which are regulated. In this case, the function Φ is integrable if and only if it satisfies the Conditions (2.4.3), (2.4.4) and (2.4.5). The following lemma shows that if Φ is Hilbert-Schmidt-valued these conditions are already satisfied, i.e. Φ is stochastically integrable. This is in line with the general integration theory for random integrands developed in [27], where the random integrands are assumed to have càdlàg trajectories.

Lemma 3.2.2. Every regulated function $\Phi\colon [0,T] \to \mathcal{L}_2(U,V)$ is stochastically integrable with respect to L.

Proof. We first show that Φ is weakly in $R([0,T];U)$. Let $t \in (0,T]$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $t_n \uparrow t$ as $n \to \infty$. Since Φ is regulated, $\Phi(t-)$ exists in $\mathcal{L}_2(U,V)$ and $\|\Phi(t_n) - \Phi(t-)\|_{\mathrm{HS}} \to 0$ as $n \to \infty$. For any $v \in V$ and $n, m \in \mathbb{N}$, we obtain

$$\|\Phi^*(t_n)v - \Phi^*(t_m)v\| \leqslant \|\Phi^*(t_n) - \Phi^*(t_m)\|_{\mathrm{op}}\| v\| \leqslant \|\Phi^*(t_n) - \Phi^*(t_m)\|_{\mathrm{HS}}\|v\| \to 0,$$

as $n \to \infty$. By completeness of U, it follows that $\Phi^*(t-)v$ exists in U. Analogously, $\Phi^*(t+)v$ exists in U for each $v \in V$ and $t \in [0,T)$ and consequently, Φ is weakly in $R([0,T];U)$. We prove the stochastic integrability of Φ by verifying Conditions (2.4.3), (2.4.4) and (2.4.5). To verify (2.4.3), let $v_n \to 0$ weakly in V. The operator $\Phi^*(t)$, being Hilbert-Schmidt, is compact for each $t \in [0,T]$ and since every compact operator maps weakly convergent sequences to strongly convergent sequences, it follows that $\Phi^*(t)v_n \to 0$ in the norm topology of U. As a is continuous, $a(0) = 0$ and it maps bounded sets to bounded sets by Lemma 3.2 in [46], Lebesgue's theorem on dominated convergence implies

$$\int_A a(\Phi^*(t)v_n)\,\mathrm{d}t \to 0 \qquad \text{as } n \to \infty \text{ for each } A \in \mathfrak{B}([0,T]).$$

Since the mapping $t \mapsto \Phi(t)$ is regulated and thus bounded, we obtain

$$\int_0^T \operatorname{tr}\big[\Phi(t)Q\Phi^*(t)\big]\,\mathrm{d}t = \int_0^T \left\|\Phi(t)Q^{\frac{1}{2}}\right\|_{\mathrm{HS}}^2 \mathrm{d}t < \infty,$$

which shows Condition (2.4.4). To prove Condition (2.4.5), note that the monotone convergence theorem guarantees

$$\sup_{n\geqslant m} \int_0^T \int_U \left(\sum_{k=m}^n \langle \Phi(t)u, h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)\,\mathrm{d}t = \int_0^T f_m(t)\,\mathrm{d}t, \tag{3.2.1}$$

where for each $m \in \mathbb{N}$ and $t \in [0,T]$ we define

$$f_m(t) := \sup_{n\geqslant m} \int_U \left(\sum_{k=m}^n \langle \Phi(t)u, h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u).$$

Let λ denote the cylindrical distribution of $L(1)$. As $\Phi(t)$ is Hilbert-Schmidt for each fixed $t \in [0,T]$, the image measure $\lambda \circ \Phi^{-1}(t)$ is a genuine infinitely divisible measure with classical Lévy measure $\mu \circ \Phi^{-1}(t)$. By the definition of Lévy measures in Hilbert spaces, we have

$$\int_V \left(\|v\|^2 \wedge 1\right) (\mu \circ \Phi^{-1}(t))(\mathrm{d}v) < \infty. \tag{3.2.2}$$

Consequently, we can apply the monotone convergence theorem and Lebesgue's theorem on dominated convergence to obtain for each $t \in [0,T]$ that

$$\begin{aligned} f_m(t) &= \sup_{n\geqslant m} \int_V \left(\sum_{k=m}^n \langle v, h_k\rangle^2 \wedge 1\right) (\mu \circ \Phi^{-1}(t))(\mathrm{d}v) \\ &= \int_V \left(\sum_{k=m}^\infty \langle v, h_k\rangle^2 \wedge 1\right) (\mu \circ \Phi^{-1}(t))(\mathrm{d}v) \to 0 \qquad \text{as } m \to \infty. \end{aligned} \tag{3.2.3}$$

Since the set $K := \overline{\{\Phi(t) : t \in [0,T]\}}$ is a compact subset of $\mathcal{L}_2(U,V)$ by Theorem 2.4.1, Proposition 2.2.2 implies that the set $\{\lambda\circ\varphi^{-1} : \varphi\in K\}$ is relatively compact in the space of probability measures on $\mathfrak{B}(V)$. Since $\lambda\circ\varphi^{-1}$ is infinitely divisible with Lévy measure $\mu\circ\varphi^{-1}$, the monotone convergence theorem and part $(c)(i)$ of the compactness criterion in Theorem 2.3.3 implies

$$\sup_{\varphi\in K}\int_{\|v\|\leqslant 1}\|v\|^2(\mu\circ\varphi^{-1})(\mathrm{d}v) = \sup_{\varphi\in K}\sum_{k=1}^{\infty}\int_{\|v\|\leqslant 1}\langle v,h_k\rangle^2\,(\mu\circ\varphi^{-1})(\mathrm{d}v) < \infty$$

while from part (b) of Theorem 2.3.3 it follows that

$$\sup_{\varphi\in K}(\mu\circ\varphi^{-1})(\{v : \|v\| > 1\}) < \infty.$$

Consequently, we obtain

$$\begin{aligned}\sup_{m\in\mathbb{N}}\sup_{t\in[0,T]} f_m(t) &\leqslant \sup_{t\in[0,T]}\int_V \|v\|^2\wedge 1(\mu\circ\Phi^{-1}(t))(\mathrm{d}v)\\ &\leqslant \sup_{\varphi\in K}\int_{\|v\|\leqslant 1}\|v\|^2(\mu\circ\varphi^{-1})(\mathrm{d}v) + \sup_{\varphi\in K}\int_{\|v\|>1}(\mu\circ\varphi^{-1})(\mathrm{d}v) < \infty. \qquad (3.2.4)\end{aligned}$$

The limit (3.2.3) and the inequality (3.2.4) enable us to apply Lebesgue's theorem in (3.2.1), which proves Condition (2.4.5). $\square$

For some $u\in U$ and $v\in V$, we define the operator $u\otimes v\colon U\to V$ by $(u\otimes v)(w) := \langle u,w\rangle v$.

Lemma 3.2.3. If $\Phi\colon [0,T]\to\mathcal{L}_2(U,V)$ is a regulated function, then $\sum\limits_{j=1}^{m} e_j\otimes\Phi(\cdot)e_j$ converges to Φ in $R\big([0,T],\mathcal{L}_2(U,V)\big)$ as $m\to\infty$.

Proof. By Theorem 2.4.1, the set $K := \overline{\{\Phi(t) : t\in[0,T]\}}$ is compact in $\mathcal{L}_2(U,V)$. By applying (**??**) we conclude

$$\sup_{t\in[0,T]}\left\|\Phi(t)-\sum_{j=1}^{m}e_j\otimes\Phi(t)e_j\right\|_{\mathrm{HS}}^2 = \sup_{t\in[0,T]}\sum_{i=1}^{\infty}\left\|\Phi(t)e_i-\sum_{j=1}^{m}\langle e_j,e_i\rangle\Phi(t)e_j\right\|^2$$

$$
\begin{aligned}
&= \sup_{t\in[0,T]} \sum_{i=m+1}^{\infty} \|\Phi(t)e_i\|^2 \\
&\leqslant \sup_{\varphi\in K} \sum_{i=m+1}^{\infty} \|\varphi e_i\|^2 \\
&\to 0 \text{ as } m \to \infty,
\end{aligned}
$$

which completes the proof. □

Lemma 3.2.4. For each regulated function $\Phi\colon [0,T] \to \mathcal{L}_2(U,V)$ there exists a sequence of partitions $\{(t_k^n)_{k=0}^{N_n} : n \in \mathbb{N}\}$ of $[0,T]$ with $\max_{0\leqslant k\leqslant N_n-1} |t_{k+1}^n - t_k^n| \to 0$ as $n \to \infty$ such that the functions

$$
\Phi_{m,n}(t) := \begin{cases} \displaystyle\sum_{j=1}^{m} e_j \otimes \Phi\left(\tfrac{t_k^n + t_{k+1}^n}{2}\right) e_j, & \text{if } t \in (t_k^n, t_{k+1}^n),\ k = 0, \dots, N_n - 1, \\ \displaystyle\sum_{j=1}^{m} e_j \otimes \Phi(t_k^n) e_j, & \text{if } t = t_k^n,\ k = 0, \dots, N_n, \end{cases} \tag{3.2.5}
$$

satisfy

$$
\lim_{m,n\to\infty} \sup_{t\in[0,T]} \|\Phi_{m,n}(t) - \Phi(t)\|_{\mathrm{HS}} = 0, \tag{3.2.6}
$$

and

$$
\lim_{m,n\to\infty} \int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t) = \int_0^T \Phi(t)\,\mathrm{d}L(t) \qquad \text{in probability.} \tag{3.2.7}
$$

Proof. Using Theorem 2.4.1(c), we can construct a sequence $\{(t_k^n)_{k=0}^{N_n} : n \in \mathbb{N}\}$ of partitions of $[0,T]$ such that $\max\limits_{0\leqslant k\leqslant N_n-1} |t_{k+1}^n - t_k^n| \to 0$ and that the functions

$$
\Phi_n(t) := \begin{cases} \Phi\left(\tfrac{t_k^n + t_{k+1}^n}{2}\right), & \text{if } t \in (t_k^n, t_{k+1}^n),\ k = 0, \dots, N_n - 1, \\ \Phi(t_k^n), & \text{if } t = t_k^n,\ k = 0, \dots, N_n, \end{cases}
$$

satisfy

$$\sup_{t\in[0,T]} \|\Phi(t)-\Phi_n(t)\|_{\mathrm{HS}} \to 0 \quad \text{ as } n\to\infty.$$

Let $\varepsilon>0$ be given. Then there exists $N\in\mathbb{N}$ such that for all $n\geqslant N$, we have

$$\sup_{t\in[0,T]} \|\Phi(t)-\Phi_n(t)\|_{\mathrm{HS}} \leqslant \frac{\varepsilon}{2}. \tag{3.2.8}$$

By Lemma 3.2.3, there exists $M>0$, such that for all $m\geqslant M$, we have

$$\sup_{t\in[0,T]} \|\Phi(t)-\sum_{j=1}^{m} e_j\otimes \Phi(t)e_j\|_{\mathrm{HS}} \leqslant \frac{\varepsilon}{2}. \tag{3.2.9}$$

Using (3.2.8) and (3.2.9) we have for all $n\geqslant N$ and $m\geqslant M$,

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\Phi(t)-\Phi_{m,n}(t)\|_{\mathrm{HS}}\\
&\qquad\leqslant \sup_{t\in[0,T]} \|\Phi(t)-\Phi_n(t)\|_{\mathrm{HS}} + \sup_{t\in[0,T]} \|\Phi_n(t)-\Phi_{m,n}(t)\|_{\mathrm{HS}}\\
&\qquad= \sup_{t\in[0,T]} \|\Phi(t)-\Phi_n(t)\|_{\mathrm{HS}} + \sup_{t\in[0,T]} \left(\sum_{k=0}^{N_n} \mathbb{1}_{\{t_k^n\}}(t) \left\|\Phi(t_k^n)-\sum_{j=1}^{m} e_j\otimes\Phi(t_k^n)e_j\right\|_{\mathrm{HS}}\right.\\
&\qquad\qquad\left. + \sum_{k=0}^{N_n-1} \mathbb{1}_{(t_k^n,t_{k+1}^n)}(t) \left\|\Phi\left(\tfrac{t_k^n+t_{k+1}^n}{2}\right) - \sum_{j=1}^{m} e_j\otimes\Phi\left(\tfrac{t_k^n+t_{k+1}^n}{2}\right)e_j\right\|_{\mathrm{HS}}\right)\\
&\qquad\leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}$$

which proves (3.2.6). Let $P_{m,n}$ denote the probability distribution of $\int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t)$. For establishing (3.2.7), it is sufficient by [26, Lemma 2.4] to show:

(i) $\left\langle \int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t) - \int_0^T \Phi(t)\,\mathrm{d}L(t), v\right\rangle \to 0$ in probability for all $v\in V$;

(ii) $\{P_{m,n} : m,n \in \mathbb{N}\}$ is relatively compact in $\mathcal{M}(V)$.

As $\Phi^*_{m,n}(\cdot)v$ converges uniformly to $\Phi^*(\cdot)v$ for each $v \in V$ due to (3.2.6), Lemma 2.4.2 implies

$$\left\langle \int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t) - \int_0^T \Phi(t)\,\mathrm{d}L(t), v \right\rangle = \int_0^T \big(\Phi^*_{m,n}(t) - \Phi^*(t)\big)v\,\mathrm{d}L(t) \to 0$$

in probability which establishes (i). To prove (ii), we define the set

$$K_1 := \left\{ \sum_{j=1}^m e_j \otimes \varphi e_j : m \in \mathbb{N} \cup \{\infty\}, \varphi \in K \right\},$$

where $K := \overline{\{\Phi(t) : t \in [0,T]\}}$. Since K is a compact subset of $\mathcal{L}_2(U,V)$ by Theorem 2.4.1, it follows that K_1 is closed and bounded. By applying (2.1.2) we obtain

$$\begin{aligned}
\lim_{N\to\infty} \sup_{\psi \in K_1} \sum_{k=N+1}^{\infty} \|\psi e_k\|^2 &= \lim_{N\to\infty} \sup_{\varphi\in K} \sup_{m\in\mathbb{N}\cup\{\infty\}} \sum_{k=N+1}^{\infty} \Big\| \sum_{j=1}^m \langle e_j, e_k\rangle \varphi e_j \Big\|^2 \\
&= \lim_{N\to\infty} \sup_{\varphi\in K} \sum_{k=N+1}^{\infty} \|\varphi e_k\|^2 = 0.
\end{aligned}$$

Similarly applying (2.1.3), we have

$$\begin{aligned}
\lim_{N\to\infty} \sup_{\psi \in K_1} \sum_{k=N+1}^{\infty} \|\psi^* h_k\|^2 &= \lim_{N\to\infty} \sup_{\varphi\in K} \sup_{m\in\mathbb{N}\cup\{\infty\}} \sum_{k=N+1}^{\infty} \Big\| \sum_{j=1}^m \langle \varphi e_j, h_k\rangle e_j \Big\|^2 \\
&\leqslant \lim_{N\to\infty} \sup_{\varphi\in K} \sum_{k=N+1}^{\infty} \|\varphi^* h_k\|^2 = 0,
\end{aligned}$$

which shows by Theorem 2.1.1 that K_1 is a compact subset of $\mathcal{L}_2(U,V)$. Proposition 2.2.2 guarantees that the set $\{\lambda \circ \psi^{-1} : \psi \in K_1\}$ is relatively compact in the space of probability measures on $\mathfrak{B}(V)$, where λ is the cylindrical distribution of $L(1)$. By (2.4.2), for each $v \in V$

and each $m, n \in \mathbb{N}$, we have

$$\left\langle \int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t), v \right\rangle = \int_0^T \Phi^*_{m,n}(t) v\,\mathrm{d}L(t) = \sum_{k=0}^{N_n-1} \big(L(t^n_{k+1}) - L(t^n_k)\big)\left((\psi^n_{m,k})^* v\right) \quad (3.2.10)$$

where $\psi^n_{m,k} := \sum_{j=1}^m e_j \otimes \Phi\left(\frac{t^n_k + t^n_{k+1}}{2}\right) e_j$. Since for each k, the operator $\psi^n_{m,k}$ is Hilbert-Schmidt, we obtain a classical random variable $X^n_{m,k} \colon \Omega \to V$ satisfying

$$\big(L(t^n_{k+1}) - L(t^n_k)\big)\left((\psi^n_{m,k})^* h\right) = \langle X^n_{m,k}, h\rangle \qquad \text{for all } h \in V. \quad (3.2.11)$$

The distribution of $X^n_{m,k}$ is given by $(\lambda \circ (\psi^n_{m,k})^{-1})^{*(t^n_{k+1} - t^n_k)}$ and consequently it follows by (3.2.10) and (3.2.11) that

$$P_{m,n} = (\lambda \circ (\psi^n_{m,0})^{-1})^{*(t^n_1 - t^n_0)} * \cdots * (\lambda \circ (\psi^n_{m,N_n-1})^{-1})^{*(t^n_{N_n} - t^n_{N_n-1})}.$$

Finally, since $\psi^n_{m,k}$ is in the compact set K_1 for each $k \in \{0, \dots, N_n\}$, Lemma 5.4 in [27] implies (ii). $\square$

Lemma 3.2.5. The mapping

$$J \colon R\big([0,T]; L^2_\eta(S;U)\big) \to R\big([0,T]; \mathcal{L}_2(U, L^2_\eta(S;\mathbb{R}))\big), \qquad J(f)(t)u = \langle u, f(t)(\cdot)\rangle,$$

is a well defined isometric isomorphism.

Proof. For each $t \in [0,T]$ and $f \in R\big([0,T]; L^2_\eta(S;U)\big)$, the map $J(f)(t)$ defines a linear and continuous operator from U to $L^2_\eta(S;\mathbb{R})$ and satisfies, by monotone convergence theorem,

$$\|J(f)(t)\|^2_{\mathrm{HS}} = \sum_{j=1}^\infty \|\langle e_j, f(t)(\cdot)\rangle\|^2_{L^2_\eta(S;\mathbb{R})}$$

$$= \int_S \sum_{j=1}^{\infty} \langle e_j, f(t)(s)\rangle^2 \, \eta(\mathrm{d}s)$$

$$= \|f(t)\|^2_{L^2_\eta(S;U)}. \tag{3.2.12}$$

As $t \mapsto f(t)$ is regulated, the isometry (3.2.12) shows by a Cauchy argument that $t \mapsto J(f)(t)$ is regulated. Consequently, J is a well defined linear isometry and it is left to show that J is surjective.

For this purpose, let Φ be in $R\big([0,T]; \mathcal{L}_2(U, L^2_\eta(S;\mathbb{R}))\big)$. We define

$$f\colon [0,T] \to L^2_\eta(S;U), \qquad f(t)(\cdot) := \sum_{j=1}^{\infty} \left(\Phi(t)e_j\right)(\cdot) e_j,$$

where the series converges in $L^2_\eta(S;U)$ by

$$\begin{aligned}
\int_S \Big\| \sum_{j=1}^{\infty} \left(\Phi(t)e_j\right)(s) e_j \Big\|^2 \,\mathrm{d}s &= \int_S \sum_{j=1}^{\infty} \left|\left(\Phi(t)e_j\right)(s)\right|^2 \,\mathrm{d}s \\
&= \sum_{j=1}^{\infty} \int_S \left|\left(\Phi(t)e_j\right)(s)\right|^2 \,\mathrm{d}s \\
&= \sum_{j=1}^{\infty} \|\Phi(t)e_j\|^2_{L^2_\eta(S;\mathbb{R})} \\
&= \|\Phi(t)\|^2_{\mathrm{HS}} < \infty.
\end{aligned}$$

As $\|f(t)\|_{L^2_\eta(S;U)} = \|\Phi(t)\|_{\mathcal{L}_2(U, L^2_\eta(S;\mathbb{R}))}$, the function $t \mapsto f(t)$ is regulated and satisfies

$$(J(f)(t))(u) = \sum_{j=1}^{\infty} \Phi(t)e_j(\cdot)\langle u, e_j\rangle = \sum_{j=1}^{\infty} \Phi(t)\big(\langle u, e_j\rangle e_j\big)(\cdot) = \Phi(t)\,(u)\,(\cdot),$$

which completes the proof. □

Lemma 3.2.6. Let $g\colon S \times [0,T] \to U$ be a function such that the map $t \to g(s,t)$ is regulated for η-almost all $s \in S$, and $\{(t_k^n)_{k=0}^{N_n} : n \in \mathbb{N}\}$ be a sequence of partitions of $[0,T]$ with $\max_{0 \leqslant k \leqslant N_n - 1} |t_{k+1}^n - t_k^n| \to 0$. Then the functions $g_{m,n} \colon S \times [0,T] \to U$ defined by

$$g_{m,n}(s,t) := \begin{cases} \displaystyle\sum_{j=1}^{m} \left\langle e_j, g\left(s, \tfrac{t_k^n + t_{k+1}^n}{2}\right)\right\rangle e_j & \text{if } t \in (t_k^n, t_{k+1}^n), k = 0, \dots, N_n - 1, \\ \displaystyle\sum_{j=1}^{m} \langle e_j, g(s, t_k^n)\rangle e_j, & \text{if } t = t_k^n, k = 0, \dots, N_n, \end{cases} \tag{3.2.13}$$

satisfy for η-almost all $s \in S$ that

$$\|g_{m,n}(s,t) - g(s,t)\| \to 0 \quad \text{for Lebesgue-almost all } t \in [0,T].$$

Proof. For each $n \in \mathbb{N}$, define $g_n \colon S \times [0,T] \to U$ by

$$g_n(s,t) := \sum_{k=0}^{N_n - 1} \mathbb{1}_{(t_k^n, t_{k+1}^n)}(t) g\left(s, \frac{t_k^n + t_{k+1}^n}{2}\right) + \sum_{k=0}^{N_n - 1} \mathbb{1}_{\{t_k^n\}}(t) g(s, t_k^n).$$

Let $s \in S$ be such that $g(s,\cdot)$ is regulated. Then the set $A_s \subseteq [0,T]$ of discontinuities of $g(s,\cdot)$ has Lebesgue measure 0 and we claim that for each $t \in A_s^c$ it follows that

$$\lim_{n\to\infty} \|g_n(s,t) - g(s,t)\| = 0. \tag{3.2.14}$$

Since $t \in A_s^c$ is a point of continuity of $g(s,\cdot)$, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|g(s,r) - g(s,t)\| < \varepsilon \text{ whenever } |t - r| < \delta.$$

By assumption on the sequence of partitions $\{(t_k^n)_{k=0}^{N_n} : n \in \mathbb{N}\}$, we can find $M \in \mathbb{N}$ such that for all $n \geqslant M$, we have

$$\max_{0 \leqslant k \leqslant N_n} |t_{k+1}^n - t_k^n| < \delta.$$

Consequently, for all $n \geqslant M$, we obtain

$$\|g_n(s,t) - g(s,t)\| = \sum_{k=0}^{N_n-1} \mathbb{1}_{(t_k^n, t_{k+1}^n)}(t) \left\| g\left(s, \frac{t_k^n + t_{k+1}^n}{2}\right) - g(s,t) \right\| < \varepsilon,$$

which proves our claim. The set $\overline{\{g(s,t) : t \in [0,T]\}}$ is compact in U by Theorem 2.4.1. The compactness criterion in Hilbert spaces implies

$$\begin{aligned} \sup_{t\in[0,T]} \left\| \sum_{j=1}^{m} \langle g(s,t), e_j\rangle e_j - g(s,t) \right\|^2 &= \sup_{t\in[0,T]} \sum_{j=m+1}^{\infty} \langle g(s,t), e_j\rangle^2 \\ &\to 0 \quad \text{as } m \to \infty. \end{aligned} \tag{3.2.15}$$

By using (3.2.14) and (3.2.15) we obtain for each $t \in A_s^c$ that

$$\begin{aligned} &\|g_{m,n}(s,t) - g(s,t)\| \\ &\qquad \leqslant \|g_{m,n}(s,t) - g_n(s,t)\| + \|g_n(s,t) - g(s,t)\| \\ &\qquad = \sum_{k=0}^{N_n-1} \mathbb{1}_{(t_k^n, t_{k+1}^n)}(t) \left\| \sum_{j=1}^{m} \left\langle g\left(s, \tfrac{t_k^n+t_{k+1}^n}{2}\right), e_j \right\rangle e_j - g\left(s, \tfrac{t_k^n+t_{k+1}^n}{2}\right) \right\| \\ &\qquad\quad + \sum_{k=0}^{N_n} \mathbb{1}_{\{t_k^n\}}(t) \left\| \sum_{j=1}^{m} \langle g(s,t_k^n), e_j\rangle e_j - g(s,t_k^n) \right\| + \|g_n(s,t) - g(s,t)\| \\ &\qquad \leqslant \sup_{r\in[0,T]} \left\| \sum_{j=1}^{m} \langle g(s,r), e_j\rangle e_j - g(s,r) \right\| + \|g_n(s,t) - g(s,t)\| \\ &\qquad \to 0 \qquad \text{as } m,n \to \infty, \end{aligned}$$

which completes the proof. □

Let $(A, \mathcal{A}, \sigma)$ be a finite measure space and (E,d) be a complete metric space. By $L_\sigma^0(A;E)$ we denote the space of the equivalence classes of all separably-valued, measurable functions (equivalently, strongly measurable functions) from A to E. As before, $L^0_\sigma(A;E)$ is a complete metric space with respect to the metric

$$\rho(f,g) := \int_A (d(f(x),g(x)) \wedge 1)\,\sigma(\mathrm{d}x), \tag{3.2.16}$$

and convergence in the metric ρ is equivalent to convergence in σ-measure. The following lemma can be proved in the same way as in Lemma III.11.16 in [23] by replacing L^p-norms for $p \geqslant 1$ by the corresponding metrics as defined in (3.2.16). The proof is given only for the sake of completeness.

Lemma 3.2.7. Let $(A_1, \mathcal{A}_1, \sigma_1)$ and $(A_2, \mathcal{A}_2, \sigma_2)$ be two finite measure spaces and V be a separable Hilbert space. Then

$$L^0_{\sigma_1}(A_1; L^0_{\sigma_2}(A_2;V)) \cong L^0_{\sigma_1\otimes\sigma_2}(A_1 \times A_2; V).$$

In particular, the isomorphism is given such that for each $\mathcal{A}_1$-measurable function $F\colon A_1 \to L^0_{\sigma_2}(A_2;V)$, there corresponds an $\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable function $f\colon A_1 \times A_2 \to V$ such that for σ_1-almost all $x \in A_1$, we have $F(x) = f(x,\cdot)$ in $L^0_{\sigma_2}(A_2;V)$ and conversely.

Proof. Let $F\colon A_1 \to L^0_{\sigma_2}(A_2;V)$ be strongly measurable. Then we can find a sequence of simple functions $(F_n)_{n\in\mathbb{N}}$ with $F_n(x) \to F(x)$ in the metric of $L^0_{\sigma_2}(A_2;V)$ and

$$F_n(x) = \sum_{k=1}^{M_n} h_k^{(n)} \mathbb{1}_{C_k^{(n)}}(x), \quad x \in A_1,$$

where $(C_k^{(n)})$ is a partition of measurable subsets of A_1 and $h_k^{(n)} \in L^0_{\sigma_2}(A_2;V)$. Define the functions

$$f_n : A_1 \times A_2 \to V, \qquad f_n(x,y) := \sum_{k=1}^{M_n} h_k^{(n)}(y) \mathbb{1}_{C_k^{(n)}}(x).$$

Then f_n is $\sigma_1 \otimes \sigma_2$ measurable and $f_n(x,\cdot) = F_n(x)$ for $x \in A_1$. Using Fubini's theorem and Lebesgue's theorem on dominated convergence, we have

$$\begin{aligned}
&\int_{A_1 \times A_2} (\|f_n(x,y) - f_m(x,y)\| \wedge 1)\, (\sigma_1 \otimes \sigma_2)(\mathrm{d}(x,y)) \\
&= \int_{A_1} \int_{A_2} (\|f_n(x,y) - f_m(x,y)\| \wedge 1)\, \sigma_2(\mathrm{d}y)\sigma_1(\mathrm{d}x) \\
&= \int_{A_1} \rho_2(F_n(x), F_m(x))\sigma_1(\mathrm{d}x) \\
&\to 0 \text{ as } m,n \to \infty,
\end{aligned}$$

where ρ_2 denotes the metric in $L^0_{\sigma_2}(A_2;V)$. Consequently, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^0_{\sigma_1\otimes\sigma_2}(A_1 \times A_2;V)$ and thus there exists a $\sigma_1 \otimes \sigma_2$-measurable function f such that

$$f_n \to f \text{ in } L^0_{\sigma_1\otimes\sigma_2}(A_1 \times A_2;V).$$

Fubini theorem implies by passing to a subsequence that for almost all $x \in A_1$, we have $f_n(x,\cdot) \to f(x,\cdot)$ in $L^0_{\sigma_2}(A_2;V)$. Hence $f(x,\cdot) = F(x)$ for almost all $x \in A_1$.

Conversely, let $f\colon A_1 \times A_2 \to V$ be a measurable function. Then we can find a sequence of simple functions $f_n\colon A_1 \times A_2 \to V$, each of which is a finite linear combination of functions of the form $\mathbb{1}_{A\times B}(\cdot)v$ with $A \in \mathcal{A}_1$, $B \in \mathcal{A}_2$, $v \in V$ such that $f_n \to f$ in $\sigma_1 \otimes \sigma_2$-measure and passing to a subsequence we may assume that $f(x,y) = \lim_{n\to\infty} f_n(x,y)$ for almost all $(x,y) \in A_1 \times A_2$. Then by Fubini's theorem, for almost all $x \in A_1$,

$$f_n(x,y) \to f(x,y) \qquad \text{as } n \to \infty, \text{ for almost all } y \in A_2.$$

Define the mappings $F_n(x) := f_n(x,\cdot)$ and $F(x) := f(x,\cdot)$ from A_1 to $L^0_{\sigma_2}(A_2;V)$. Clearly the functions F_n being $\mathcal{A}_1$-simple functions are measurable. An application of Lebesgue's theorem implies that for almost all $x \in A_1$, the sequence $F_n(x)$ converges to $F(x)$ in the metric of $L^0_{\sigma_2}(A_2; V)$. Then F being an almost everywhere limit of $\mathcal{A}_1$-measurable functions is strongly measurable and hence belongs to $L^0_{\sigma_1}(A_1; L^0_{\sigma_2}(A_2; V))$. □

Remark 3.2.8. As a consequence of the above Lemma, we get the following embedding

$$L^0_P(\Omega; L^2_\eta(S; U)) \hookrightarrow L^0_P(\Omega; L^0_\eta(S; U)) \cong L^0_{\eta\otimes P}(S \times \Omega; U). \tag{3.2.17}$$

Proof of Theorem 3.2.1. Lemma 3.2.5 guarantees that the mapping

$$\Phi\colon [0,T] \to \mathcal{L}_2(U, L^2_\eta(S; \mathbb{R})), \qquad \Phi(t)u := \langle u, g(\cdot, t)\rangle,$$

is well defined and regulated. Let $\Phi_{m,n}$ denote the functions defined in (3.2.5) for $V = L^2_\eta(S; \mathbb{R})$. Lemma 3.2.4 together with Lemma 3.2.7 and the remark that follows imply, upon passing to a subsequence, that for $(\eta \otimes P)$-almost all $(s, \omega) \in S \times \Omega$ we have

$$\left(\left(\int_0^T \Phi(t)\,\mathrm{d}L(t)\right)(\omega)\right)(s) = \lim_{m,n\to\infty} \left(\left(\int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t)\right)(\omega)\right)(s). \tag{3.2.18}$$

For each $h \in L^2_\eta(S; \mathbb{R})$, we obtain by (2.4.2) that

$$\begin{aligned}
&\left\langle \int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t), h \right\rangle_{L^2_\eta(S;\mathbb{R})} \\
&\qquad = \int_0^T \Phi^*_{m,n}(t) h\,\mathrm{d}L(t) \\
&\qquad = \sum_{k=0}^{N_n-1} \big(L(t^n_{k+1}) - L(t^n_k)\big) \left(\sum_{j=1}^m \left\langle e_j, \Phi^*\left(\frac{t^n_k + t^n_{k+1}}{2}\right) h \right\rangle e_j \right) \\
&\qquad = \sum_{k=0}^{N_n-1} \sum_{j=1}^m \left\langle \Phi\left(\frac{t^n_k + t^n_{k+1}}{2}\right) e_j, h \right\rangle_{L^2_\eta(S;\mathbb{R})} \big(L(t^n_{k+1}) - L(t^n_k)\big)(e_j)
\end{aligned}$$

$$= \left\langle \sum_{k=0}^{N_n-1} \sum_{j=1}^{m} \Phi\left(\frac{t_k^n + t_{k+1}^n}{2}\right) e_j \big(L(t_{k+1}^n) - L(t_k^n)\big)(e_j), h \right\rangle_{L^2_\eta(S;\mathbb{R})}.$$

Therefore, for η-almost all $s \in S$, we have

$$\begin{aligned}
\left(\int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t)\right)(s) &= \left(\sum_{k=0}^{N_n-1} \sum_{j=1}^{m} \Phi\left(\frac{t_k^n + t_{k+1}^n}{2}\right) e_j \big(L(t_{k+1}^n) - L(t_k^n)\big)(e_j)\right)(s) \\
&= \sum_{k=0}^{N_n-1} \sum_{j=1}^{m} \left(\Phi\left(\frac{t_k^n + t_{k+1}^n}{2}\right) e_j\right)(s) \big(L(t_{k+1}^n) - L(t_k^n)\big)(e_j) \\
&= \sum_{k=0}^{N_n-1} \big(L(t_{k+1}^n) - L(t_k^n)\big) \left(\sum_{j=1}^{m} \left(\Phi\left(\frac{t_k^n + t_{k+1}^n}{2}\right) e_j\right)(s) e_j\right) \\
&= \int_0^T g_{m,n}(s,t)\,\mathrm{d}L(t), \qquad (3.2.19)
\end{aligned}$$

where $g_{m,n}$ denotes the function defined in (3.2.13). For each $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}
&E\left[\exp\left(i\alpha \int_0^T \big(g_{m,n}(s,t) - g(s,t)\big)\,\mathrm{d}L(t)\right)\right] \\
&\qquad = \exp\left(\int_0^T \Psi\big(\alpha\left(g_{m,n}(s,t) - g(s,t)\right)\big)\,\mathrm{d}t\right), \qquad (3.2.20)
\end{aligned}$$

where Ψ denotes the Lévy symbol of L. Note that

$$\sup_{t\in[0,T]} \|g_{m,n}(s,t) - g(s,t)\|^2 \leqslant 4 \sup_{t\in[0,T]} \|g(s,t)\|^2 < \infty.$$

Since Ψ is continuous and maps bounded sets to bounded sets according to Lemma 3.2 in [46], it follows by Lebesgue's theorem on dominated convergence and Lemma 3.2.6 that

$$\lim_{m,n\to\infty} \int_0^T \Psi\left(\alpha(g_{m,n}(s,t) - g(s,t))\right)\,\mathrm{d}t = 0.$$

Consequently, we deduce from (3.2.20) that for η-almost all $s \in S$,

$$\lim_{m,n\to\infty} \int_0^T g_{m,n}(s,t)\,\mathrm{d}L(t) = \int_0^T g(s,t)\,\mathrm{d}L(t) \qquad \text{in probability.} \tag{3.2.21}$$

Comparing limits in (3.2.18) and (3.2.21) by means of (3.2.19), we obtain that for η-almost all $s \in S$, we have

$$\left(\int_0^T g(s,t)\,\mathrm{d}L(t)\right)(\omega) = \left(\left(\int_0^T \Phi(t)\,\mathrm{d}L(t)\right)(\omega)\right)(s) \quad \text{for } P\text{-almost all } \omega \in \Omega. \tag{3.2.22}$$

By (3.2.21) and Lemma 3.2.7, the left hand side in (3.2.22) is $\mathcal{S} \otimes \mathcal{F}$ measurable, as well as the right hand side due to (3.2.18). A further application of Fubini's theorem implies for P-almost all $\omega \in \Omega$, that

$$\left(\int_0^T g(s,t)\,\mathrm{d}L(t)\right)(\omega) = \left(\left(\int_0^T \Phi(t)\,\mathrm{d}L(t)\right)(\omega)\right)(s) \quad \text{for } \eta\text{-almost all } s \in S.$$

Since the right side is in $L^2_\eta(S;\mathbb{R})$, we obtain that for almost all $\omega \in \Omega$, the map $s \to \left(\int_0^T g(s,t)\,\mathrm{d}L(t)\right)(\omega)$ belongs to $L^2_\eta(S;\mathbb{R})$. By integrating both sides and denoting by 1 the function in $L^2_\eta(S;\mathbb{R})$ which constantly equals one, we obtain by (2.4.2) that

$$\begin{aligned}
\int_S \left(\int_0^T g(s,t)\,\mathrm{d}L(t)\right)(\omega)\,\eta(\mathrm{d}s) &= \int_S \left(\left(\int_0^T \Phi(t)\,\mathrm{d}L(t)\right)(\omega)\right)(s)\,\eta(\mathrm{d}s) \\
&= \left\langle \left(\int_0^T \Phi(t)\,\mathrm{d}L(t)\right)(\omega), 1 \right\rangle_{L^2_\eta(S;\mathbb{R})} \\
&= \left(\int_0^T \Phi^*(t)1\,\mathrm{d}L(t)\right)(\omega) \\
&= \left(\int_0^T \int_S g(s,t)\eta(\mathrm{d}s)\,\mathrm{d}L(t)\right)(\omega),
\end{aligned}$$

where the last equality is obtained by noting that for any $f \in L^2_\eta(S;\mathbb{R})$, $u \in U$,

$$\langle \Phi(t)u, f\rangle = \langle \langle u, g(s,\cdot)\rangle, f\rangle = \int_S \langle u, g(s,t)\rangle f(s)\eta(\mathrm{d}s) = \left\langle u, \int_S g(s,t)f(s)\eta(\mathrm{d}s)\right\rangle,$$

which implies that $\Phi^*(t)f = \int_S g(s,t)f(s)\eta(\mathrm{d}s)$. This completes the proof. $\square$

Chapter 4

Cauchy Problem

In this chapter we prove the existence and uniqueness of the stochastic Cauchy problem by establishing the equivalence of the mild solution and the weak solution. This Chapter is based on joint work with my supervisor Prof. Markus Riedle and is close to Section 4 and some part of Section 5 in the preprint [31].

4.1 Introduction

We consider the following stochastic Cauchy problem driven by a cylindrical Lévy process L in a separable Hilbert space U:

$$\begin{aligned} \mathrm{d}Y(t) &= AY(t)\,\mathrm{d}t + B\,\mathrm{d}L(t) \qquad \text{for all } t \in [0,T], \\ Y(0) &= Y_0, \end{aligned} \tag{4.1.1}$$

where A is a generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a separable Hilbert space V, $B\colon U \to V$ is a linear and continuous operator and the initial condition Y_0 is a V-valued $\mathcal{F}_0$-measurable random variable.

In case L is a cylindrical Brownian motion, the concept of weak solution is defined in [20] and the existence and uniqueness of a weak solution is established. Their definition requires weak solutions to have almost surely Bochner integrable paths. In case of Banach spaces, a similar definition is used in [56]. However, as it is known that the solution of (4.1.1) may exhibit highly irregular paths, the requirement of Bochner integrable paths is too restrictive in our situation. A weaker condition requires only that the paths $t \mapsto \langle Y(t), A^*v\rangle$ are integrable for $v \in \mathcal{D}(A^*)$; see [13], [38] and [58] . We will impose a slightly stronger condition but which is still weaker than Bochner integrability of the paths.

Definition 4.1.1. A V-valued stochastic process $(Y(t) : t \geqslant 0)$ is called *weakly Bochner regular* if $t \mapsto \langle Y(t), g(t)\rangle$ is integrable on $[0, T]$ for each $g \in C([0, T]; V)$ and for every sequence $(g_n)_{n\in\mathbb{N}} \subseteq C([0, T]; V)$ with $\|g_n\|_\infty \to 0$ as $n \to \infty$, we have

$$\int_0^T \langle Y(s), g_n(s)\rangle \,\mathrm{d}s \to 0 \qquad \text{in probability } n \to \infty.$$

If the stochastic process Y has Bochner integrable paths on $[0, T]$, then Y is also weakly Bochner regular by the following estimate

$$\int_0^T |\langle Y(s), g_n(s)\rangle| \,\mathrm{d}s \leqslant \|g_n\|_\infty \int_0^T \|Y(s)\| \,\mathrm{d}s.$$

Definition 4.1.2. A V-valued, progressively measurable stochastic process $(Y(t) : t \in [0, T])$ is called a *weak solution* of the stochastic Cauchy problem (4.1.1) if Y is weakly Bochner regular and satisfies for every $v \in \mathcal{D}(A^*)$ and $t \in [0, T]$, P-almost surely, that

$$\langle Y(t), v\rangle = \langle Y_0, v\rangle + \int_0^t \langle Y(s), A^*v\rangle \,\mathrm{d}s + L(t)(B^*v). \tag{4.1.2}$$

4.2 Stochastic Convolution

In this section we define the stochastic convolution process and prove that it is stochastically continuous.

Suppose that the map $s \to T(s)B$ is stochastically integrable in $[0,T]$. Then by Lemma 6.2 in [46], for each $t \in [0,T]$, the map $s \to T(t-s)B$ is also stochastically integrable on $[0,t]$. For each $t \in [0,T]$, we define

$$Y_A(t) := \int_0^t T(t-s)B \, \mathrm{d}L(s). \tag{4.2.1}$$

The process $(Y_A(t) : t \in [0,T])$ is called the *stochastic convolution* process. The characteristic function $\varphi_{Y_A(t)} \colon V \to \mathbb{C}$ of $Y_A(t)$ is given by

$$\varphi_{Y_A(t)}(v) = \exp\left(\int_0^t \Psi(B^*T^*(t-s)v) \, \mathrm{d}s\right) = \exp\left(\int_0^t \Psi(B^*T^*(s)v) \, \mathrm{d}s\right)$$

Consequently, we have

$$Y_A(t) \stackrel{d}{=} \int_0^t T(s)B \, \mathrm{d}L(s). \tag{4.2.2}$$

Let ν_t denote the probability distribution of $Y_A(t)$. Then it follows from Lemma 5.4 in [46] that ν_t is an infinitely divisible probability measure on $\mathfrak{B}(V)$ with characteristics (c_t, S_t, ξ_t) given, for all $v \in V$, by

$$\begin{aligned}
\langle c_t, v\rangle &= \int_0^t a(B^*T^*(s)v) \, \mathrm{d}s + \int_V \langle h, v\rangle \left(\mathbb{1}_{B_V}(h) - \mathbb{1}_{B_{\mathbb{R}}}(\langle h, v\rangle)\right) \xi_t(\mathrm{d}h), \\
\langle v, S_t v\rangle &= \int_0^t \langle B^*T^*(s)v, QB^*T^*(s)v\rangle \, \mathrm{d}s, \\
\xi_t &= (\mathrm{leb} \otimes \mu) \circ \chi_t^{-1} \quad \text{on } \mathcal{Z}(V),
\end{aligned}$$

where $\chi_t \colon [0,t] \times U \to V$ is defined by $\chi_t(s,u) := T(s)Bu$.

Theorem 4.2.1. *The stochastic convolution process $(Y_A(t) : t \in [0,T])$ is stochastically continuous.*

Proof. By [26, Lemma 2.4], it is enough to show that

(i) $\big(\langle Y_A(t), v\rangle : t \in [0,T]\big)$ is stochastically continuous for each $v \in V$;

(ii) $\{\nu_t : t \in [0,T]\}$ is relatively compact in $\mathcal{M}(V)$.

Proof of (i): for every $t \in [0,T]$, $v \in V$ and $\varepsilon > 0$, we have by (2.4.2) that

$$\begin{aligned}
&|\langle Y_A(t+\varepsilon), v\rangle - \langle Y_A(t), v\rangle| \\
&\quad = \left| \int_0^{t+\varepsilon} B^*T^*(t+\varepsilon-s)v \,\mathrm{d}L(s) - \int_0^t B^*T^*(t-s)v \,\mathrm{d}L(s) \right| \\
&\quad \leqslant \left| \int_0^t B^*T^*(t-s)(T^*(\varepsilon)v - v) \,\mathrm{d}L(s) \right| + \left| \int_t^{t+\varepsilon} B^*T^*(t+\varepsilon-s)v \,\mathrm{d}L(s) \right|. \qquad (4.2.3)
\end{aligned}$$

Define the random variables

$$I_1(\varepsilon) := \int_0^t B^*T^*(t-s)(T^*(\varepsilon)v - v) \,\mathrm{d}L(s), \quad I_2(\varepsilon) := \int_t^{t+\varepsilon} B^*T^*(t+\varepsilon-s)v \,\mathrm{d}L(s).$$

The random variable $I_1(\varepsilon)$ has the characteristic function $\varphi_{1,\varepsilon} \colon \mathbb{R} \to \mathbb{C}$ given by

$$\begin{aligned}
\varphi_{1,\varepsilon}(\beta) &= \exp\left(\int_0^t \Psi\big(\beta B^*T^*(t-s)(T^*(\varepsilon)v - v)\big) \,\mathrm{d}s \right) \\
&= \exp\left(\int_0^t \Psi\big(\beta B^*T^*(s)(T^*(\varepsilon)v - v)\big) \,\mathrm{d}s \right).
\end{aligned}$$

By using standard properties of the semigroup we obtain

$$\sup_{s \in [0,T]} \|\beta B^*T^*(s)(T^*(\varepsilon)v - v)\| \to 0 \qquad \text{as } \varepsilon \to 0, \qquad (4.2.4)$$

which implies $\varphi_{1,\varepsilon}(\beta) \to 1$ for all $\beta \in \mathbb{R}$ due to Lemma 5.1 in [46]. Thus, $I_1(\varepsilon)$ converges to 0 in probability as $\varepsilon \to 0$. The characteristic function $\varphi_{2,\varepsilon}\colon \mathbb{R} \to \mathbb{C}$ of the random variable $I_2(\varepsilon)$ obeys

$$\begin{aligned}\varphi_{2,\varepsilon}(\beta) &= \exp\left(\int_t^{t+\varepsilon} \Psi(\beta B^*T^*(t+\varepsilon-s)v)\,\mathrm{d}s\right)\\ &= \exp\left(\int_0^{\varepsilon} \Psi(\beta B^*T^*(s)v)\,\mathrm{d}s\right) \to 1 \text{ as } \varepsilon \to 0.\end{aligned}$$

Consequently, we obtain that $I_2(\varepsilon) \to 0$ in probability. The arguments above show by (4.2.3) that $\langle Y_A(t+\varepsilon), v\rangle \to \langle Y_A(t), v\rangle$ in probability as $\varepsilon \to 0$. Analogously, we show that $\langle Y_A(t-\varepsilon), v\rangle \to \langle Y_A(t), v\rangle$ in probability as $\varepsilon \to 0$. As before, using the semigroup property, we have

$$\begin{aligned}&|\langle Y_A(t-\varepsilon), v\rangle - \langle Y_A(t), v\rangle|\\ &\quad= \left|\int_0^{t-\varepsilon} B^*T^*(t-\varepsilon-s)v\,\mathrm{d}L(s) - \int_0^t B^*T^*(t-s)v\,\mathrm{d}L(s)\right|\\ &\quad\leqslant \left|\int_0^{t-\varepsilon} B^*T^*(t-\varepsilon-s)(T^*(\varepsilon)v-v)\,\mathrm{d}L(s)\right| + \left|\int_{t-\varepsilon}^t B^*T^*(t-s)v\,\mathrm{d}L(s)\right|. \qquad (4.2.5)\end{aligned}$$

Define the random variables

$$I_3(\varepsilon) := \int_0^{t-\varepsilon} B^*T^*(t-\varepsilon-s)(T^*(\varepsilon)v-v)\,\mathrm{d}L(s), \quad I_4(\varepsilon) := \int_{t-\varepsilon}^t B^*T^*(t-s)v\,\mathrm{d}L(s).$$

Since the random variable $I_4(\varepsilon)$ has the same characteristic function as that of the random variable $I_2(\varepsilon)$, it follows that $I_4(\varepsilon) \to 0$ in probability as $\varepsilon \to 0$. The characteristic function $\varphi_{3,\varepsilon}\colon \mathbb{R} \to \mathbb{C}$ of the random variable $I_3(\varepsilon)$ is given by

$$\varphi_{3,\varepsilon}(\beta) = \exp\left(\int_0^{t-\varepsilon} \Psi(\beta B^*T^*(t-\varepsilon-s)(T^*(\varepsilon)v-v))\,\mathrm{d}s\right)$$

$$= \exp\left(\int_0^{t-\varepsilon} \Psi(\beta B^* T^*(s)(T^*(\varepsilon)v - v))\,\mathrm{d}s\right)$$

It follows by (4.2.4) and Lemma 5.1 in [46] that

$$\begin{aligned}
\left|\int_0^{t-\varepsilon} \Psi(\beta B^* T^*(s)(T^*(\varepsilon)v - v))\,\mathrm{d}s\right| &\leqslant \int_0^{t-\varepsilon} |\Psi(\beta B^* T^*(s)(T^*(\varepsilon)v - v))|\,\mathrm{d}s \\
&\leqslant \int_0^{T} |\Psi(\beta B^* T^*(s)(T^*(\varepsilon)v - v))|\,\mathrm{d}s \\
&\to 0 \qquad \text{as } \varepsilon \to 0,
\end{aligned}$$

which implies $\varphi_{3,\varepsilon}(\beta) \to 1$ for all $\beta \in \mathbb{R}$. As a consequence $I_3(\varepsilon) \to 0$ in probability as $\varepsilon \to 0$ and hence $\langle Y(t-\varepsilon), v\rangle \to \langle Y(t), v\rangle$ in probability, which yields Property (i).

Proof of (ii): Let $\tilde{\nu}_t$ denote the infinitely divisible probability measure with characteristics $(0, S_t, \xi_t)$. Theorem 2.3.3 guarantees that the set $\{\tilde{\nu}_t : t \in [0,T]\}$ is relatively compact if and only if the set $\{\xi_t : t \in [0,T]\}$ restricted to the complement of any neighbourhood of the origin is relatively compact in $\mathcal{M}(V)$ and the operators $T_t\colon V \to V$ defined by

$$\langle T_t v, v\rangle := \langle S_t v, v\rangle + \int_{\|h\|\leqslant 1} \langle v, h\rangle^2\,\xi_t(\mathrm{d}h)$$

satisfy

$$\sup_{t\in[0,T]} \sum_{k=1}^{\infty} \langle T_t h_k, h_k\rangle < \infty \qquad \text{and} \qquad \lim_{N\to\infty} \sup_{t\in[0,T]} \sum_{k=N}^{\infty} \langle T_t h_k, h_k\rangle = 0. \tag{4.2.6}$$

For a set A in the cylindrical algebra $\mathcal{Z}(V)$, we have

$$\xi_t(A) = \int_0^t \int_U \mathbb{1}_A(T(s)Bu)\mu(\mathrm{d}u)\,\mathrm{d}s \leqslant \int_0^T \int_U \mathbb{1}_A(T(s)Bu)\mu(\mathrm{d}u)\,\mathrm{d}s = \xi_T(A).$$

Since $\mathfrak{B}(V)$ is the sigma algebra generated by $\mathcal{Z}(V)$ and $\mathcal{Z}(V)$ is closed under intersection, by

Caratheodary's extension theorem and uniqueness of extension we conclude $\xi_t \leqslant \xi_T$ on $\mathfrak{B}(V)$ for all $t \in [0,T]$. Let ξ_t^c denote the restriction of ξ_t to the complement of a neighbourhood of the origin $V_1 \subset V$. Since ξ_T^c is a Radon measure by [33, Prop 1.1.3], there exists for each $\varepsilon > 0$ a compact set $K \subseteq V_1$ such that $\xi_T^c(K^c) \leqslant \varepsilon$. Consequently, we obtain $\xi_t^c(K^c) \leqslant \xi_T^c(K^c) \leqslant \varepsilon$ for all $t \in [0,T]$, which shows by Prokhorov's theorem that $\{\xi_t : t \in [0,T]\}$ restricted to the complement of any neighbourhood of the origin is relatively compact in $\mathcal{M}(V)$.

The stochastic integrability of $s \mapsto T(s)B$ implies by (2.4.4) and Lebesgue's theorem that

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sum_{k=N}^{\infty} \langle S_t h_k, h_k\rangle \, \mathrm{d}s = \lim_{N\to\infty} \int_0^T \sum_{k=N}^{\infty} \langle T(s)BQB^*T^*(s)h_k, h_k\rangle \, \mathrm{d}s = 0. \tag{4.2.7}$$

Condition (2.4.5) of stochastic integrability implies

$$\begin{aligned}
\sup_{t\in[0,T]} \sum_{k=N}^{\infty} \int_{\|h\|\leqslant 1} \langle h_k, h\rangle^2 \xi_t(\mathrm{d}h) &\leqslant \sup_{t\in[0,T]} \sup_{m\geqslant N} \int_V \left(\sum_{k=N}^{m} \langle h_k, h\rangle^2 \wedge 1 \right) \xi_t(\mathrm{d}h) \\
&= \sup_{t\in[0,T]} \sup_{m\geqslant N} \int_0^t \int_U \left(\sum_{k=N}^{m} \langle h_k, T(s)Bu\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\, \mathrm{d}s \\
&= \sup_{m\geqslant N} \int_0^T \int_U \left(\sum_{k=N}^{m} \langle h_k, T(s)Bu\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\to 0 \qquad \text{as } N \to \infty.
\end{aligned} \tag{4.2.8}$$

The limits (4.2.7) and (4.2.8) show that the second condition in (4.2.6) is satisfied. The first condition in (4.2.6) follows by, using monotone convergence theorem,

$$\begin{aligned}
\sup_{t\in[0,T]} \sum_{k=1}^{\infty} \langle T_t h_k, h_k\rangle &= \sup_{t\in[0,T]} \sum_{k=1}^{\infty} \left(\langle S_t h_k, h_k\rangle + \int_{\|h\|\leqslant 1} \langle h_k, h\rangle^2 \xi_t(\mathrm{d}h) \right) \\
&= \sup_{t\in[0,T]} \sum_{k=1}^{\infty} \int_0^t \langle T(s)BQB^*T^*(s)h_k, h_k\rangle \, \mathrm{d}s + \sup_{t\in[0,T]} \int_{\|h\|\leqslant 1} \|h\|^2 \xi_t(\mathrm{d}h) \\
&= \int_0^T \operatorname{tr}\left(T(s)BQB^*T^*(s)\right) \mathrm{d}s + \int_{\|h\|\leqslant 1} \|h\|^2 \xi_T(\mathrm{d}h)
\end{aligned}$$

$$< \infty,$$

and hence we conclude that $\{\tilde{\nu}_t : t \in [0,T]\}$ is relatively compact. Let $\{\tilde{\nu}_{t_n}\}_{n\in\mathbb{N}}$ be a weakly convergent subsequence. Without any restriction we can assume that there exists $t \in [0,T]$ such that $t_n \to t$. For the characteristic functions $\varphi_{\nu_{t_n}}$ of ν_{t_n} we obtain

$$\begin{aligned}
&|\varphi_{\nu_{t_n}}(v) - \varphi_{\nu_t}(v)| \\
&\quad = \left|\exp\left(\int_0^{t_n} \Psi(B^*T^*(t_n - s)v)\,\mathrm{d}s\right) - \exp\left(\int_0^{t} \Psi(B^*T^*(t - s)v)\,\mathrm{d}s\right)\right| \\
&\quad = \left|\exp\left(\int_t^{t_n} \Psi(B^*T^*(s)v)\,\mathrm{d}s\right) - 1\right| \left|\exp\left(\int_0^{t} \Psi(B^*T^*(s)v)\,\mathrm{d}s\right)\right| \\
&\quad \leqslant \left(\exp\left|\int_t^{t_n} \Psi(B^*T^*(s)v)\,\mathrm{d}s\right| - 1\right) \exp\left(\int_0^{T} |\Psi(B^*T^*(s)v)|\,\mathrm{d}s\right). \qquad (4.2.9)
\end{aligned}$$

Since Ψ maps bounded sets to bounded sets, we obtain for each $\delta > 0$ that

$$\sup_{\|v\|<\delta} \left|\int_t^{t_n} \Psi(B^*T^*(s)v)\,\mathrm{d}s\right| \to 0 \quad \text{as } n \to \infty,$$

which implies by (4.2.9) that

$$\sup_{\|v\|<\delta} |\varphi_{\nu_{t_n}}(v) - \varphi_{\nu_t}(v)| \to 0 \quad \text{as } n \to \infty.$$

As $\tilde{\nu}_{t_n} = \nu_{t_n} * \delta_{-c_{t_n}}$, Theorem 2.3.8 in [33] implies that $\{\nu_{t_n}\}$ converges weakly, which completes the proof of Property (ii). $\square$

4.3 Existence of weak solution

Theorem 4.3.1. *If the mapping $s \mapsto T(s)B$ is stochastically integrable on $[0,T]$ with respect to L, then*

$$Y(t) = T(t)Y_0 + \int_0^t T(t-s)B \, \mathrm{d}L(s), \quad t \in [0,T], \tag{4.3.1}$$

is a weak solution of the stochastic Cauchy problem (4.1.1).

Remark 4.3.2. The process $(Y(t) : t \geqslant 0)$ defined by (4.3.1) is called the mild solution of (4.1.1). In other words, this theorem implies that the mild solution is a weak solution of (4.1.1).

Proof of Theorem 4.3.1. We first observe that Y is a weak solution of (4.1.1) with initial condition $Y(0) = Y_0$ if and only if the process $(\tilde{Y}(t) : t \in [0,T])$ defined by $\tilde{Y}(t) := Y(t) - T(t)Y_0$ is a weak solution of (4.1.1) with initial condition $Y(0) = 0$. Suppose Y is a weak solution of (4.1.1). It easily follows that the process $\tilde{Y}$ is progressively measurable and weakly Bochner regular. Then for every $v \in \mathcal{D}(A^*)$, we obtain by (4.1.2) that

$$\begin{aligned}
\left\langle \tilde{Y}(t), v \right\rangle &= \langle Y(t) - T(t)Y_0, v \rangle \\
&= \langle Y_0, v \rangle + \int_0^t \langle Y(s), A^*v \rangle \, \mathrm{d}s + L(t)(B^*v) - \langle T(t)Y_0, v \rangle \\
&= \langle Y_0 - T(t)Y_0, v \rangle + \int_0^t \langle Y(s), A^*v \rangle \, \mathrm{d}s + L(t)(B^*v) \\
&= -\left\langle A \int_0^t T(s)Y_0 \, \mathrm{d}s, v \right\rangle + \int_0^t \langle Y(s), A^*v \rangle \, \mathrm{d}s + L(t)(B^*v) \\
&= -\int_0^t \langle T(s)Y_0, A^*v \rangle \, \mathrm{d}s + \int_0^t \langle Y(s), A^*v \rangle \, \mathrm{d}s + L(t)(B^*v) \\
&= \int_0^t \left\langle \tilde{Y}(s), A^*v \right\rangle \, \mathrm{d}s + L(t)(B^*v),
\end{aligned}$$

which proves that $\tilde{Y}$ is a weak solution of (4.1.1) with initial condition $Y(0) = 0$. A similar computation shows the converse assertion. Therefore, without loss of generality we can assume that $Y_0 = 0$.

Lemma 6.2 in [46] guarantees that the map $r \mapsto T(s-r)B$ is stochastically integrable on $[0, s]$ for each $s \in (0, T]$. Thus, we can define

$$Y(s) := \int_0^s T(s-r)B \, \mathrm{d}L(r) \qquad \text{for all } s \in [0, T].$$

We first show that Y is weakly Bochner regular. Let g be in $C([0,T]; V)$ and define

$$f \colon [0, T] \times [0, T] \to U, \qquad f(s, r) = \mathbb{1}_{[0,s]}(r) B^* T^*(s-r) g(s). \tag{4.3.2}$$

By using (2.4.2) we conclude for all $s \in [0, T]$ that

$$\langle Y(s), g(s) \rangle = \int_0^s B^* T^*(s-r) g(s) \, \mathrm{d}L(r) = \int_0^T f(s, r) \, \mathrm{d}L(r). \tag{4.3.3}$$

For fixed $s \in [0, T]$ the map $r \mapsto f(s, r)$ is regulated. By defining $m := \sup_{s \in [0,T]} \|B^* T^*(s)\|_{\mathrm{op}}^2$, we obtain for $\varepsilon > 0$ and $r \in [0, T - \varepsilon]$ that

$$\begin{aligned}
&\|f(\cdot, r+\varepsilon) - f(\cdot, r)\|_{L^2([0,T];U)}^2 \\
&= \int_0^T \|\mathbb{1}_{[r+\varepsilon, T]}(s) B^* T^*(s-r-\varepsilon) g(s) - \mathbb{1}_{[r,T]}(s) B^* T^*(s-r) g(s)\|^2 \, \mathrm{d}s \\
&= \int_{r+\varepsilon}^T \|B^* T^*(s-r-\varepsilon)(\mathrm{Id} - T^*(\varepsilon)) g(s)\|^2 \, \mathrm{d}s + \int_r^{r+\varepsilon} \|B^* T^*(s-r) g(s)\|^2 \, \mathrm{d}s. \\
&\leqslant m \int_0^T \|(\mathrm{Id} - T^*(\varepsilon)) g(s)\|^2 \, \mathrm{d}s + \varepsilon m \|g\|_\infty^2 \\
&\to 0 \qquad \text{as } \varepsilon \to 0,
\end{aligned}$$

which shows that f is right continuous. In a similar way, we establish that $r \to f(\cdot, r)$ is left continuous. Indeed, for $\varepsilon > 0$ and $r \in [\varepsilon, T]$ we have

$$
\begin{aligned}
&\|f(\cdot, r-\varepsilon) - f(\cdot, r)\|^2_{L^2([0,T];U)} \\
&= \int_0^T \|\mathbb{1}_{[r-\varepsilon,T]}(s) B^* T^*(s-r+\varepsilon) g(s) - \mathbb{1}_{[r,T]}(s) B^* T^*(s-r) g(s)\|^2 \, \mathrm{d}s \\
&= \int_r^T \|B^* T^*(s-r)(T^*(\varepsilon) - \mathrm{Id}) g(s)\|^2 \, \mathrm{d}s + \int_{r-\varepsilon}^r \|B^* T^*(s-r+\varepsilon) g(s)\|^2 \, \mathrm{d}s. \\
&\leqslant m \int_0^T \|(T^*(\varepsilon) - \mathrm{Id}) g(s)\|^2 \, \mathrm{d}s + \varepsilon m \|g\|^2_\infty \\
&\to 0 \qquad \text{as } \varepsilon \to 0,
\end{aligned}
$$

Thus, we can apply Theorem 3.2.1 to conclude by using (4.3.3) that the mapping $s \mapsto \langle Y(s), g(s) \rangle$ is square-integrable on $[0, T]$ and

$$
\int_0^T \langle Y(s), g(s) \rangle \, \mathrm{d}s = \int_0^T \int_0^s B^* T^*(s-r) g(s) \, \mathrm{d}L(r) \, \mathrm{d}s = \int_0^T \int_r^T B^* T^*(s-r) g(s) \, \mathrm{d}s \, \mathrm{d}L(r). \tag{4.3.4}
$$

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $C([0,T]; V)$ with $\|g_n\|_\infty \to 0$. By (4.3.4) and Lemma 5.4 in [46], the Lévy symbol of the infinitely divisible random variable $\int_0^T \langle Y(s), g_n(s) \rangle \, \mathrm{d}s$ is given by

$$
\Phi_n \colon \mathbb{R} \to \mathbb{C}, \qquad \Phi_n(\beta) = \int_0^T \Psi \left(\int_r^T \beta B^* T^*(s-r) g_n(s) \, \mathrm{d}s \right) \mathrm{d}r,
$$

where $\Psi \colon U \to \mathbb{C}$ is the Lévy symbol of L. As Ψ is continuous and maps bounded sets to bounded sets according to Lemma 3.2 and Lemma 5.1 of [46], a repeated application of Lebesgue's theorem implies $\Phi_n(\beta) \to 0$ for every $\beta \in \mathbb{R}$, which proves that Y is weakly Bochner regular.

Taking $T = t$ and $g(s) = A^* v$ for every $s \in [0, t]$ in the definition of f in (4.3.2), we can apply Theorem 3.2.1 to obtain for each $v \in \mathcal{D}(A^*)$ that

$$\begin{aligned}
\int_0^t \langle Y(s), A^*v\rangle \,\mathrm{d}s &= \int_0^t \left(\int_0^s B^*T^*(s-r)A^*v \,\mathrm{d}L(r) \right) \mathrm{d}s \\
&= \int_0^t \left(\int_r^t B^*T^*(s-r)A^*v \,\mathrm{d}s \right) \mathrm{d}L(r) \\
&= \int_0^t \left(B^* \int_r^t \frac{\mathrm{d}}{\mathrm{d}s}(T^*(s-r)v)\,\mathrm{d}s \right) \mathrm{d}L(r) \\
&= \int_0^t \left(B^*T^*(t-r)v - B^*T^*(0)v\right) \mathrm{d}L(r) \\
&= \int_0^t B^*T^*(t-r)v \,\mathrm{d}L(r) - L(t)(B^*v) \\
&= \langle Y(t), v\rangle - L(t)(B^*v),
\end{aligned}$$

which shows (4.1.2). Theorem 4.2.1 guarantees that the stochastic process $\big(\int_0^t T(t-r)B\,\mathrm{d}L(r) : t \in [0,T]\big)$ is stochastically continuous and since it is also adapted, it has a progressively measurable modification by Proposition 3.6 in [20] which completes the proof. □

Example 4.3.3. In this and the next example we set $V = U$, $B = \mathrm{Id}$ and assume that there exist $\lambda_k \geqslant 0$ with $\lambda_k \to \infty$ as $k \to \infty$ such that

$$T^*(t)e_k = e^{-\lambda_k t}e_k \qquad \text{for all } t \in [0,T],\ k \in \mathbb{N}. \tag{4.3.5}$$

In the literature, e.g. [12], [34], [35] and [39], often cylindrical Lévy processes of the following form are considered:

$$L(t)u := \sum_{k=1}^{\infty} \langle e_k, u\rangle \sigma_k \ell_k(t) \qquad \text{for all } t \in [0,T],\ u \in U, \tag{4.3.6}$$

where $(\ell_k)_{k\in\mathbb{N}}$ is a sequence of independent, symmetric, real valued Lévy processes with characteristics $(0, 0, \mu_k)$ and $(\sigma_k)_{k\in\mathbb{N}}$ is a real valued sequence such that the series in (4.3.6) converges in $L_P^0(\Omega;\mathbb{R})$. By using (2.4.5) we obtain that $T(\cdot)$ is stochastically integrable with respect to L if and only if

$$\sum_{k=1}^{\infty} \int_0^T \int_{\mathbb{R}} \left(e^{-2\lambda_k s} |\sigma_k \beta|^2 \wedge 1 \right) \mu_k(\mathrm{d}\beta)\, \mathrm{d}t < \infty; \tag{4.3.7}$$

see Corollary 6.3 in [46]. For example, if $(\ell_k)_{k\in\mathbb{N}}$ is a family of independent, identically distributed, standardised, symmetric α-stable processes with $\alpha \in (0,2)$, one easily computes that $T(\cdot)$ is stochastically integrable w.r.t. L if and only if

$$\sum_{k=1}^{\infty} \frac{|\sigma_k|^\alpha}{\lambda_k} < \infty. \tag{4.3.8}$$

This result on the existence of a weak solution of the stochastic Cauchy problem (4.1.1) coincides with the result in [41].

Example 4.3.4. We assume the same setting as in Example 4.3.3 but model L as the canonical α-stable cylindrical Lévy process defined in Example 2.3.10. It is shown in Theorem 4.1 in [42] that a semigroup $(T(t))_{t\geqslant 0}$ satisfying the spectral decomposition (4.3.5) is stochastically integrable with respect to L if and only if

$$\int_0^T \|T(s)\|_{\mathrm{HS}}^\alpha \,\mathrm{d}s < \infty. \tag{4.3.9}$$

In the work [14], the authors consider the stochastic Cauchy problem in Banach spaces driven by a subordinated cylindrical Brownian motion (as defined in Example 2.3.11), a slightly more general noise than the canonical α-stable cylindrical Lévy process. As the approach in [14] relies on embedding the underlying space U in a larger space, the derived conditions are less explicit than (4.3.9) and only sufficient.

4.4 Uniqueness of weak solution

To prove uniqueness of the solution, we use the same approach as in [20] for which we need the following integration by parts formula.

Lemma 4.4.1. If $g\colon [0,T] \to U$ is a function of the form $g(t) = \tau(t)u$ for $u \in U$ and $\tau \in C^1([0,T];\mathbb{R})$, then

$$\int_0^T g(s)\,\mathrm{d}L(s) = -\int_0^T L(s)(g'(s))\,\mathrm{d}s + L(T)(g(T)).$$

Proof. For a sequence $\{(t_k^n)_{k=0}^{N_n} : n \in \mathbb{N}\}$ of partitions of $[0,T]$ with $\max_{0\leqslant k\leqslant N_n-1} |t_{k+1}^n - t_k^n| \to 0$ as $n \to \infty$ define the simple functions

$$g_n\colon [0,T] \to U, \qquad g_n(t) := \sum_{k=0}^{N_n-1} g(t_k^n)\mathbb{1}_{[t_k^n, t_{k+1}^n)}(t) + \mathbb{1}_{\{T\}}(t)g(T).$$

As g_n converges to g uniformly on $[0,T]$, Lemma 2.4.2 implies

$$\int_0^T g_n(s)\,\mathrm{d}L(s) \to \int_0^T g(s)\,\mathrm{d}L(s) \qquad \text{in probability.} \tag{4.4.1}$$

On the other hand, P-almost surely we obtain

$$\begin{aligned}
&\int_0^T g_n(s)\,\mathrm{d}L(s) \qquad\qquad (4.4.2)\\
&\quad= \sum_{k=0}^{N_n-1} \left(L(t_{k+1}^n) - L(t_k^n)\right)(\tau(t_k^n)u)\\
&\quad= \sum_{k=0}^{N_n-1} \tau(t_k^n)\left(L(t_{k+1}^n) - L(t_k^n)\right)(u)\\
&\quad= -\sum_{k=0}^{N_n-1} \left(\tau(t_{k+1}^n) - \tau(t_k^n)\right)L(t_{k+1}^n)(u) + \sum_{k=0}^{N_n-1} \left(\tau(t_{k+1}^n)L(t_{k+1}^n)(u) - \tau(t_k^n)L(t_k^n)(u)\right)
\end{aligned}$$

$$= -\sum_{k=0}^{N_n-1} \big(\tau(t_{k+1}^n) - \tau(t_k^n)\big) L(t_{k+1}^n)(u) + \tau(T)L(T)(u). \tag{4.4.3}$$

Applying the mean value theorem, we obtain for some $\xi_k^n \in (t_k^n, t_{k+1}^n)$ that

$$\begin{aligned}
&\sum_{k=0}^{N_n-1} \big(\tau(t_{k+1}^n) - \tau(t_k^n)\big)\, L(t_{k+1}^n)(u) \\
&= \sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n) L(t_{k+1}^n)(u) \\
&= \sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n) L(\xi_k^n)(u) - \sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n)\big(L(\xi_k^n)(u) - L(t_{k+1}^n)(u)\big).
\end{aligned} \tag{4.4.4}$$

As the map $s \mapsto \tau'(s)L(s)u$ has only a countable number of discontinuities, it is Riemann integrable and we obtain

$$\lim_{n\to\infty} \sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n) L(\xi_k^n)(u) = \int_0^T L(s)(u\tau'(s))\,\mathrm{d}s. \tag{4.4.5}$$

To show that the second term in (4.4.4) approaches 0 we define

$$M_k^n := \sup_{s\in[t_k^n, t_{k+1}^n]} L(s)u, \qquad m_k^n := \inf_{s\in[t_k^n, t_{k+1}^n]} L(s)u.$$

Riemann integrability of the map $s \mapsto L(s)u$ implies

$$\begin{aligned}
&\left|\sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n)\big(L(\xi_k^n)(u) - L(t_{k+1}^n)(u)\big)\right| \\
&\qquad \leqslant \sum_{k=0}^{N_n-1} \left|\tau'(\xi_k^n)\right| \left|t_{k+1}^n - t_k^n\right| \left|L(\xi_k^n)(u) - L(t_{k+1}^n)(u)\right| \\
&\qquad \leqslant \|\tau'\|_\infty \sum_{k=0}^{N_n-1} \left|t_{k+1}^n - t_k^n\right| \left|M_k^n - m_k^n\right|
\end{aligned}$$

$$\to 0 \qquad \text{as} \quad n \to \infty. \tag{4.4.6}$$

Taking the limit in (4.4.2) by applying (4.4.4), (4.4.5) and (4.4.6) and comparing it to the limit in (4.4.1) completes the proof. □

Theorem 4.4.2. *If there exists a weak solution Y of the stochastic Cauchy problem* (4.1.1) *then the mapping $s \mapsto T(s)B$ is stochastically integrable on $[0,T]$ with respect to L and Y is given by*

$$Y(t) = T(t)Y_0 + \int_0^t T(t-s)B \,\mathrm{d}L(s).$$

Proof. With the same reason as in the proof of Theorem 4.3.1, we can assume that $Y_0 = 0$. For every $v \in \mathcal{D}(A^*)$ and $s \in [0,T]$ we have P-a.s. that

$$\langle Y(s), v\rangle = \int_0^s \langle Y(r), A^*v\rangle \,\mathrm{d}r + L(s)(B^*v). \tag{4.4.7}$$

Fix $t \in [0,T]$. Since Y is progressively measurable, both sides in (4.4.7) are $\mathfrak{B}([0,t]) \otimes \mathcal{F}$-measurable and hence by Fubini's theorem, we have P-a.s.

$$\langle Y(s), v\rangle = \int_0^s \langle Y(r), A^*v\rangle \,\mathrm{d}r + L(s)(B^*v), \tag{4.4.8}$$

for almost all $s \in (0,t)$. Let f be in $C^1([0,t];\mathbb{R})$ and v in $\mathcal{D}(A^*)$. By using (4.4.8) and applying the integration by parts formula in Lemma 4.4.1 to $g(\cdot) = f(\cdot)B^*v$ and the classical integration by parts formula for Lebesgue integrals we obtain

$$\begin{aligned}\int_0^t f'(s)\langle Y(s), v\rangle \,\mathrm{d}s &= \int_0^t f'(s)\left(\int_0^s \langle Y(r), A^*v\rangle \,\mathrm{d}r\right)\mathrm{d}s + \int_0^t f'(s)L(s)(B^*v)\,\mathrm{d}s \\ &= f(t)\int_0^t \langle Y(s), A^*v\rangle \,\mathrm{d}s - \int_0^t f(s)\langle Y(s), A^*v\rangle \,\mathrm{d}s\end{aligned}$$

$$+ f(t)L(t)(B^*v) - \int_0^t f(s)B^*v \, \mathrm{d}L(s).$$

Rearranging the terms and using (4.4.8), we obtain by defining $F(\cdot) = f(\cdot)v$ that

$$\langle Y(t), F(t) \rangle = \int_0^t \langle Y(s), F'(s) + A^*F(s) \rangle \, \mathrm{d}s + \int_0^t B^*F(s) \, \mathrm{d}L(s). \tag{4.4.9}$$

For $v \in \mathcal{D}(A^{*2})$, the function $G := T^*(t - \cdot)v$ is in $C^1([0,t]; \mathcal{D}(A^*))$. Due to Lemma 8.4 in [56], we can find a sequence $F_n \in \operatorname{span}\{f(\cdot)w : f \in C^1([0,t]; \mathbb{R}),\ w \in \mathcal{D}(A^*)\}$ such that F_n converges to G in $C^1([0,t]; \mathcal{D}(A^*))$. Then $F_n' + A^*F_n \to 0$ in $C([0,t]; V)$. The weak Bochner regularity implies that

$$\int_0^t \langle Y(s), F_n'(s) + A^*F_n(s) \rangle \, \mathrm{d}s \to 0 \quad \text{in probability.}$$

Moreover, since B^*F_n converges to B^*G in $C([0,t]; U)$, Lemma 2.4.2 implies

$$\int_0^t B^*F_n(s) \, \mathrm{d}L(s) \to \int_0^t B^*G(s) \, \mathrm{d}L(s) \qquad \text{in probability.}$$

Consequently, (4.4.9) holds for F replaced by G, which gives

$$\langle Y(t), v \rangle = \int_0^t B^*T^*(t-s)v \, \mathrm{d}L(s) \qquad \text{for all } v \in \mathcal{D}(A^{*2}).$$

Since $\mathcal{D}(A^{*2})$ is dense in V, for any $v \in V$, we can find a sequence $\{v_n\}$ in $\mathcal{D}(A^{*2})$ with $v_n \to v$ as $n \to \infty$. Since $B^*T^*(t - \cdot)v_n \to B^*T^*(t - \cdot)v$ in $C([0,t]; U)$ it follows from Lemma 2.4.2 that

$$\lim_{n\to\infty} \int_0^t B^*T^*(t-s)v_n \, \mathrm{d}L(s) = \int_0^t B^*T^*(t-s)v \, \mathrm{d}L(s) \qquad \text{in probability,}$$

and hence P-a.s.

$$\langle Y(t), v\rangle = \int_0^t B^* T^*(t-s) v \, \mathrm{d}L(s) \qquad \text{for all } v \in V.$$

This establishes the stochastic integrability of $s \mapsto T(s)B$ on $[0, T]$ and for each $t \in [0, T]$, we obtain

$$Y(t) = \int_0^t T(t-s) B \, \mathrm{d}L(s). \tag{4.4.10}$$

□

Remark 4.4.3. As a consequence of the above theorem we also obtain the uniqueness (up to modification) of the weak solution. Indeed, if Y_1 and Y_2 are two weak solutions of (4.1.1), then both satisfy (4.4.10), that is, for each $t \in [0, T]$, we have

$$Y_1(t) = \int_0^t T(t-s) B \, \mathrm{d}L(s) = Y_2(t),$$

which shows that Y_1 and Y_2 are modifications of each other.

Chapter 5

Properties of Solutions

In this Chapter we discuss some properties of the solution of the stochastic Cauchy problem (4.1.1). The Chapter is based on joint work with my supervisor and follows most part in Section 5 of the preprint [31].

We recall that in Chapter 4 we have proved that if $(Y(t) : t \in [0,T])$ is the weak solution of the stochastic Cauchy problem (4.1.1), then it is given by

$$Y(t) = T(t)Y_0 + Y_A(t), \qquad t \in [0,T], \tag{5.0.1}$$

where $(Y_A(t) : t \in [0,T])$ is the stochastic convolution process defined by

$$Y_A(t) = \int_0^t T(t-s)B \, \mathrm{d}L(s).$$

Various specific examples of the stochastic Cauchy problem (4.1.1) were observed in the literature in which the solution Y exists but does not have a modification $\tilde{Y}$ with scalarly càdlàg paths; see e.g. [12], [34] and [39]. Even the weaker property that the real valued process $(\langle Y(t), v\rangle : t \in [0,T])$ has a modification with càdlàg paths for each $v \in V$ can be verified only in a few specific examples. However, our stochastic Fubini Theorem 3.2.1 immediately implies that this real valued stochastic process $(\langle Y(t), v\rangle : t \in [0,T])$ has square-integrable trajectories:

Theorem 5.0.1. *If $(Y(t) : t \in [0,T])$ is the weak solution of the stochastic Cauchy problem (4.1.1), then for every $v \in V$, P-a.s.*

$$\int_0^T \langle Y(t), v\rangle^2 \,\mathrm{d}t < \infty.$$

Proof. Recall that in the proof of Theorem 4.3.1 we have shown that the function

$$f\colon [0,T] \times [0,T] \to U, \qquad f(s,t) = \mathbb{1}_{[0,t]}(s) B^* T^*(t-s)v,$$

satisfies the assumption of Theorem 3.2.1, which guarantees that $\langle Y_A(\cdot), v\rangle = \int_0^T f(s,\cdot)\,\mathrm{d}L(s)$ defines a random variable in $L^2_{\mathrm{leb}}([0,T];\mathbb{R})$. Hence P-a.s.

$$\int_0^T \langle Y(t), v\rangle^2 \,\mathrm{d}t \leqslant 2\int_0^T \langle T(t)Y_0, v\rangle^2 \,\mathrm{d}t + 2\int_0^T \langle Y_A(t), v\rangle^2 \,\mathrm{d}t < \infty,$$

which completes the proof. □

Theorem 5.0.2. *The weak solution $(Y(t) : t \in [0,T])$ of the stochastic Cauchy problem (4.1.1) is stochastically continuous.*

Proof. Since the map $t \to T(t)Y_0$ is P-a.s. continuous, the result follows by (5.0.1) and Theorem 4.2.1. □

As mentioned in the introduction, it has been observed for specific examples of a cylindrical Lévy process, that the solution of (4.1.1) has highly irregular paths in an analytical sense. In our general setting, we state a condition in the result below which implies such highly irregular paths of the solution. This condition does not only allow a geometric interpretation of this phenomena but is also easy to verify in many examples including the ones considered in the literature.

Theorem 5.0.3. *Assume that an orthonormal basis $(h_k)_{k\in\mathbb{N}}$ of V is in $\mathcal{D}(A^*)$ and let L be a cylindrical Lévy process with cylindrical characteristics (a, Q, μ). If for all $c > 0$,*

$$\lim_{n\to\infty} \mu\left(\left\{u \in U : \sum_{k=1}^{n} \langle u, B^* h_k\rangle^2 > c^2\right\}\right) = \infty, \tag{5.0.2}$$

then there does not exist any modification $\widetilde{Y}$ of the weak solution Y of (4.1.1) *such that for each $v \in V$ the stochastic process $\left(\left\langle \widetilde{Y}(t), v\right\rangle : t \in [0,T]\right)$ has càdlàg paths.*

Remark 5.0.4. Note, that if μ is a genuine Lévy measure then Condition (5.0.2) cannot be satisfied for any constant $c > 0$. This is due to the fact that in this case, μ is a finite Radon measure on complement of every open ball around the origin; see [33].

Example 5.0.5. (continues Example 4.3.3). Assume that the cylindrical Lévy process L is given by (4.3.6). The independence of the real valued Lévy processes $(\ell_k)_{k\in\mathbb{N}}$ implies that the cylindrical Lévy measure μ has support only in $\cup_{k=1}^{\infty}\text{span}\{e_k\}$, and thus Condition (5.0.2) reduces to

$$\sum_{k=1}^{\infty} \mu\Big(\Big\{u \in U : \langle u, B^* h_k\rangle^2 > c^2\Big\}\Big) = \infty,$$

for all $c > 0$. For this special case, the conclusion of Theorem 5.0.3 has already been derived in [39].

For example, if $(\ell_k)_{k\in\mathbb{N}}$ is a family of independent, identically distributed symmetric α-stable Lévy processes, then Condition (5.0.2) is satisfied for $B = \mathrm{Id}$; see [34].

Example 5.0.6. (continues Example 4.3.4). Let L be the canonical α-stable process, introduced in Example 4.3.4. By using properties of α-stable distributions in $\mathbb{R}^n$ one calculates for each $n \in \mathbb{N}$ that

$$\mu\left(\left\{u \in U : \sum_{k=1}^{n} \langle u, h_k\rangle^2 > c^2\right\}\right) = \frac{1}{c^\alpha c_\alpha} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)},$$

where Γ denotes the Gamma function and c_α is a constant only depending on α. As the right hand side converges to ∞ as $n \to \infty$, Condition (5.0.2) is satisfied for $B = \mathrm{Id}$; see [42, Theorem 5.1].

Proof of Theorem 5.0.3. (The proof is based on ideas from [34]). For every $n \in \mathbb{N}$ and $t \in [0, T]$ define the random vectors

$$L_n(t) := \big(L(t)B^*h_1, \dots, L(t)B^*h_n\big) \text{ and } Y_n(t) := \big(\langle Y(t), h_1\rangle, \dots, \langle Y(t), h_n\rangle\big).$$

It follows from Definition 4.1.2 of a weak solution that for every $t \in [0, T]$ we have P-a.s.

$$Y_n(t) = Y_n(0) + \int_0^t \big(\langle Y(s), A^*h_1\rangle, \dots, \langle Y(s), A^*h_n\rangle\big)\, \mathrm{d}s + L_n(t).$$

Consequently, the n-dimensional processes $(Y_n(t) : t \in [0, T])$ and $(L_n(t) : t \in [0, T])$ jump at the same time by the same size, which implies

$$\sup_{t\in[0,T]} |\Delta L_n(t)|^2 = \sup_{t\in[0,T]} |\Delta Y_n(t)|^2 \leqslant 4 \sup_{t\in[0,T]} |Y_n(t)|^2,$$

where $\Delta g(t) := g(t) - g(t-)$ for càdlàg functions $g \colon [0, T] \to \mathbb{R}^n$. It follows that

$$\begin{aligned} P\left(\sup_{t\in[0,T]} \sum_{k=1}^{\infty} \langle Y(t), h_k\rangle^2 < \infty\right) &= \lim_{c\to\infty} P\left(\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \sum_{k=1}^{n} \langle Y(t), h_k\rangle^2 \leqslant \frac{1}{4}c^2\right) \\ &= \lim_{c\to\infty} \lim_{n\to\infty} P\left(\sup_{t\in[0,T]} \sum_{k=1}^{n} \langle Y(t), h_k\rangle^2 \leqslant \frac{1}{4}c^2\right) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{c\to\infty}\lim_{n\to\infty} P\left(\sup_{t\in[0,T]} |Y_n(t)|^2 \leqslant \frac{1}{4}c^2\right) \\
&\leqslant \lim_{c\to\infty}\lim_{n\to\infty} P\left(\sup_{t\in[0,T]} |\Delta L_n(t)|^2 \leqslant c^2\right) \\
&= \lim_{c\to\infty}\lim_{n\to\infty} \exp\Big(-T\mu_n\Big(\{\beta\in\mathbb{R}^n : |\beta| > c\}\Big)\Big),
\end{aligned}
$$

where μ_n denotes the Lévy measure of the $\mathbb{R}^n$-valued Lévy process L_n. Since $\mu_n = \mu\circ\pi_n^{-1}$ for $\pi_n\colon U\to\mathbb{R}^n$ and $\pi_n u = (\langle u, B^*h_1\rangle\rangle, \dots, \langle u, B^*h_n\rangle)$ due to [6, Th.2.4], we obtain

$$
P\left(\sup_{t\in[0,T]}\sum_{k=1}^{\infty}\langle Y(t), h_k\rangle^2 < \infty\right) = 0,
$$

which completes the proof by an application of Theorem 2.3 in [39]. $\square$

We continue to consider mean square continuity of the trajectories of the solution. For this purpose, we naturally require that the cylindrical Lévy process has weak second moments, i.e. $E[|L(1)u|^2] < \infty$ for all $u \in U$. In this case, by Corollary 3.12 in [6], the cylindrical Lévy process with characteristics (a, Q, μ) can be written as

$$
L(t)u = t\,\langle \tilde{a}, u\rangle + W(t)u + M(t)u \qquad \text{for all } t \geqslant 0,\ u\in U, \tag{5.0.3}
$$

where $\tilde{a}\in U$, W is a cylindrical Brownian motion with covariance operator Q and M is a cylindrical Lévy process independent of W and with characteristics $(a', 0, \mu)$. Here $a'\colon U\to\mathbb{R}$ is defined by $a'(u) := -\int_{|\beta|>1}\beta\,(\mu\circ\langle\cdot,u\rangle^{-1})(\mathrm{d}\beta)$ and $\langle\tilde{a},u\rangle = a(u) - a'(u)$ for all $u\in U$. Further, for any $u\in U$, we can write

$$
E[|L(1)u|^2] = \langle\tilde{a},u\rangle^2 + \langle Qu,u\rangle + \int_U \langle u,h\rangle^2\,\mu(\mathrm{d}h). \tag{5.0.4}
$$

It follows by (5.0.3) that for any function $f \in R([0,T];U)$, we have

$$\int_0^t f(s)\,\mathrm{d}L(s) = \int_0^t \langle \tilde{a}, f(s)\rangle \;\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}W(s) + \int_0^t f(s)\,\mathrm{d}M(s). \tag{5.0.5}$$

Example 5.0.7. Assume that L has weak second moments and $E\left[\|Y_0\|^2\right] < \infty$. If

$$\int_0^T \|T(s)B\|_{\mathrm{HS}}^2 \,\mathrm{d}s < \infty, \tag{5.0.6}$$

then there exists a weak solution $(Y(t) : t \in [0,T])$ of the Cauchy problem (4.1.1) and it satisfies $E[\|Y(t)\|^2] < \infty$ for all $t \in [0,T]$.

Proof. For showing the existence of a solution, we have to establish that $t \mapsto T(t)B$ is stochastically integrable. Conditions (2.4.3) and (2.4.4) can be verified similarly as in the proof of Lemma 3.2.2 (by replacing $\Phi(t)$ by $T(t)B$). Since L has weak second moments, the closed graph theorem guarantees that $L(t)\colon U \to L^2_P(\Omega;\mathbb{R})$ is continuous, which implies by (5.0.4) that

$$C := \sup_{\|u^*\|\leqslant 1} \int_U \langle u, u^*\rangle^2 \mu(\mathrm{d}u) \leqslant \|L(1)\|_{\mathrm{op}}^2 < \infty.$$

Consequently, Condition (2.4.5) is satisfied since

$$\begin{aligned}
&\int_0^T \int_U \left(\sum_{k=m}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\quad \leqslant \sum_{k=m}^n \int_0^T \int_U \langle u, B^*T^*(s)h_k\rangle^2 \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\quad = \sum_{k=m}^n \int_0^T \int_U \|B^*T^*(s)h_k\|^2 \left\langle u, \frac{B^*T^*(s)h_k}{\|B^*T^*(s)h_k\|} \right\rangle^2 \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\quad \leqslant \sup_{\|u^*\|\leqslant 1} \int_U \langle u, u^*\rangle^2 \,\mu(\mathrm{d}u) \sum_{k=m}^n \int_0^T \|B^*T^*(s)h_k\|^2 \,\mathrm{d}s
\end{aligned}$$

$$\to 0 \qquad \text{as } m,n\to\infty, \tag{5.0.7}$$

where we applied (5.0.6) in the last line. As the Lévy measure ξ_t of the infinitely divisible random variable $Y_A(t)$ is given by $(\mathrm{leb}\otimes\mu)\circ\chi_t^{-1}$ on $\mathcal{Z}(V)$ where $\chi_t\colon [0,t]\times U\to V$ and $\chi_t(s,u)=T(s)Bu$, we obtain by a similar calculation as in (5.0.7) that

$$\int_V \|v\|^2 \,\mathrm{d}\xi_t(v)=\sum_{k=1}^\infty \int_0^t\int_U \langle u, B^*T^*(s)h_k\rangle^2\mu(\mathrm{d}u)\,\mathrm{d}s \leqslant C\int_0^t \|B^*T^*(s)\|_{\mathrm{HS}}^2\,\mathrm{d}s<\infty.$$

Consequently, we have $E[\|Y(t)\|^2]=E[\|Y_0\|^2]+E[\|Y_A(t)\|^2]<\infty$ for all $t\in[0,T]$. ☐

Theorem 5.0.8. *Assume that L has weak second moments. If the weak solution $(Y(t):t\in[0,T])$ of the stochastic Cauchy problem* (4.1.1) *has second moments, i.e. $E[\|Y(t)\|^2]<\infty$ for all $t\in[0,T]$, then Y is continuous in mean-square, i.e. $Y\in C([0,T];L_P^2(\Omega;V))$.*

Proof. Let $\Phi\colon [0,T]\to\mathcal{L}(U,V)$ be a stochastically integrable, regulated function and $\Phi(\cdot)\tilde a$ be Pettis integrable. Then we obtain for each $t\in[0,T]$ and $G\in\mathcal{L}(V,V)$ by (5.0.5) and using the fact that W and M have mean zero and are independent:

$$\begin{aligned}
&E\left[\left\|\int_0^t G\Phi(t-s)\,\mathrm{d}L(s)\right\|^2\right]\\
&\qquad=\sum_{k=1}^\infty E\left[\left|\int_0^t \Phi^*(t-s)G^*h_k\,\mathrm{d}L(s)\right|^2\right]\\
&\qquad=\sum_{k=1}^\infty\Bigg(\left|\int_0^t \langle\tilde a,\Phi^*(s)G^*h_k\rangle\,\mathrm{d}s\right|^2+\int_0^t \langle Q\Phi^*(s)G^*h_k,\Phi^*(s)G^*h_k\rangle\,\mathrm{d}s\\
&\qquad\qquad+\int_0^t\int_U \langle u,\Phi^*(s)G^*h_k\rangle^2\,\mu(\mathrm{d}u)\,\mathrm{d}s\Bigg)\\
&\qquad=\left\|\int_0^t G\Phi(s)\tilde a\,\mathrm{d}s\right\|^2+\int_0^t\left\|G\Phi(s)Q^{1/2}\right\|_{\mathrm{HS}}^2\mathrm{d}s+\int_V\|Gv\|^2\,\eta_t(\mathrm{d}v),
\end{aligned}\tag{5.0.8}$$

where η_t is the (genuine) Lévy measure of $\int_0^t\Phi(s)\,\mathrm{d}L(s)$ and is given by $\eta_t=(\mathrm{leb}\otimes\mu)\circ\zeta_t^{-1}$

where $\xi_t \colon [0,t] \times U \to V$ is defined by $\zeta_t(s,u) = \Phi(s)u$.

Since the map $t \to T(t)Y_0$ belongs to $C([0,T]; L^2_P(\Omega; V))$, we can assume without loss of generality that $Y_0 = 0$. Theorem 4.4.2 implies

$$Y(t) = \int_0^t T(t-s)B \, \mathrm{d}L(s) \qquad \text{for all } t \in [0,T].$$

As $Y(t)$ has finite second moments it follows $\int_V \|v\|^2 \, \xi_t(\mathrm{d}v) < \infty$, where ξ_t is the (genuine) Lévy measure of $Y(t)$ and is given by $\xi_t = (\mathrm{leb} \otimes \mu) \circ \chi_t^{-1}$ where $\chi_t \colon [0,t] \times U \to V$ is defined by $\chi_t(s,u) = T(s)Bu$. For any $t \in [0,T]$ and $\varepsilon > 0$ we obtain

$$\begin{aligned}
&E[\|Y(t+\varepsilon) - Y(t)\|^2] \\
&= E\left[\left\| \int_0^t (T(t+\varepsilon-s)B - T(t-s)B) \, \mathrm{d}L(s) + \int_t^{t+\varepsilon} T(t+\varepsilon-s)B \, \mathrm{d}L(s) \right\|^2\right] \\
&\leqslant 2E\left[\left\| \int_0^t (T(\varepsilon) - \mathrm{Id})T(t-s)B \, \mathrm{d}L(s) \right\|^2\right] + 2E\left[\left\| \int_t^{t+\varepsilon} T(t+\varepsilon-s)B \, \mathrm{d}L(s) \right\|^2\right]. \quad (5.0.9)
\end{aligned}$$

By applying (5.0.8) we conclude

$$\begin{aligned}
&E\left[\left\| \int_0^t (T(\varepsilon) - \mathrm{Id})T(t-s)B \, \mathrm{d}L(s) \right\|^2\right] \\
&\qquad \leqslant t \int_0^t \|(T(\varepsilon) - \mathrm{Id})T(s)B\tilde{a}\|^2 \, \mathrm{d}s + \int_0^t \left\| (T(\varepsilon) - \mathrm{Id})T(s)BQ^{1/2} \right\|^2_{\mathrm{HS}} \mathrm{d}s \\
&\qquad\qquad + \int_V \|(T(\varepsilon) - \mathrm{Id})v\|^2 \, \xi_t(\mathrm{d}v). \quad (5.0.10)
\end{aligned}$$

Applying Lebesgue's theorem to each of the terms above shows

$$E\left[\left\| \int_0^t (T(\varepsilon) - \mathrm{Id})T(t-s)B \, \mathrm{d}L(s) \right\|^2\right] \to 0 \qquad \text{as } \varepsilon \to 0. \quad (5.0.11)$$

By a similar computation as in (5.0.8) we obtain for the second term in (5.0.9) that

$$
\begin{aligned}
&E\left[\left\|\int_t^{t+\varepsilon} T(t+\varepsilon-s)B\,\mathrm{d}L(s)\right\|^2\right] \\
&\qquad\qquad = \left\|\int_0^{\varepsilon} T(s)B\tilde{a}\,\mathrm{d}s\right\|^2 + \int_0^{\varepsilon}\left\|T(s)BQ^{1/2}\right\|_{\mathrm{HS}}^2 \mathrm{d}s \\
&\qquad\qquad\qquad + \sum_{k=1}^{\infty}\int_0^T \mathbb{1}_{[0,\varepsilon]}(s)\int_U \langle u, B^*T^*(s)h_k\rangle^2\,\mu(\mathrm{d}u)\,\mathrm{d}s. \qquad (5.0.12)
\end{aligned}
$$

The first two terms in (5.0.12) converge to 0 as $\varepsilon \to 0$. Since

$$
\sum_{k=1}^{\infty}\int_0^T \mathbb{1}_{[0,\varepsilon]}(s)\int_U \langle u, B^*T^*(s)h_k\rangle^2\mu(\mathrm{d}u)\,\mathrm{d}s \leqslant \int_V \|v\|^2\,\theta_T(\mathrm{d}v) < \infty,
$$

we can apply Lebesgue's theorem to the third term in (5.0.12) and obtain

$$
E\left[\left\|\int_t^{t+\varepsilon} T(t-s)B\,\mathrm{d}L(s)\right\|^2\right] \to 0 \qquad \text{as } \varepsilon \to 0. \qquad (5.0.13)
$$

Applying (5.0.11) and (5.0.13) to (5.0.9) shows that Y is mean-square continuous from right. Similarly, to prove left continuity, for any $t \in [0,T]$ and $\varepsilon > 0$ we obtain

$$
\begin{aligned}
&E[\|Y(t-\varepsilon)-Y(t)\|^2] \\
&= E\left[\left\|\int_0^{t-\varepsilon}(T(t-\varepsilon-s)B - T(t-s)B)\,\mathrm{d}L(s) + \int_{t-\varepsilon}^{t} T(t-s)B\,\mathrm{d}L(s)\right\|^2\right] \\
&\leqslant 2E\left[\left\|\int_0^{t-\varepsilon}(T(\varepsilon)-\mathrm{Id})T(t-\varepsilon-s)B\,\mathrm{d}L(s)\right\|^2\right] + 2E\left[\left\|\int_{t-\varepsilon}^{t} T(t-s)B\,\mathrm{d}L(s)\right\|^2\right].
\end{aligned}
\qquad (5.0.14)
$$

For the second term on the right side in (5.0.14), we can use the same arguments as in (5.0.12) and (5.0.13). For the first term, by the same computation as in (5.0.8) we conclude

$$
\begin{aligned}
&E\left[\left\|\int_0^{t-\varepsilon}(T(\varepsilon)-\mathrm{Id})T(t-\varepsilon-s)B\,\mathrm{d}L(s)\right\|^2\right]\\
&\quad\leqslant t\int_0^{t-\varepsilon}\|(T(\varepsilon)-\mathrm{Id})T(t-\varepsilon-s)B\tilde{a}\|^2\,\mathrm{d}s+\int_0^{t-\varepsilon}\left\|(T(\varepsilon)-\mathrm{Id})T(t-\varepsilon-s)BQ^{1/2}\right\|_{\mathrm{HS}}^2\,\mathrm{d}s\\
&\qquad\qquad+\sum_{k=1}^{\infty}\int_0^{t-\varepsilon}\int_U\langle u,B^*T^*(t-\varepsilon-s)h_k\rangle^2\,\mu(\mathrm{d}u)\,\mathrm{d}s\\
&\quad\leqslant t\int_0^{t}\|(T(\varepsilon)-\mathrm{Id})T(s)B\tilde{a}\|^2\,\mathrm{d}s+\int_0^{t}\left\|(T(\varepsilon)-\mathrm{Id})T(s)BQ^{1/2}\right\|_{\mathrm{HS}}^2\,\mathrm{d}s\\
&\qquad\qquad+\int_V\|(T(\varepsilon)-\mathrm{Id})v\|^2\,\xi_t(\mathrm{d}v),
\end{aligned}
$$

which is the same as the right side in (5.0.10) and the left continuity follows analogously, which completes the proof. □

We now discuss the flow property and Markov property of the solution of the stochastic Cauchy problem (4.1.1). For this purpose we assume that $t\mapsto T(t)B$ is stochastically integrable and define for $0\leqslant s\leqslant t\leqslant T$ the mapping

$$\Phi_{s,t}\colon V\times\Omega\to V,\qquad \Phi_{s,t}(v)=T(t-s)v+\int_s^t T(t-r)B\,\mathrm{d}L(r).$$

Theorem 5.0.9. *Let $(Y(t):t\in[0,T])$ be the weak solution of (4.1.1). Then we have:*

(a) *the family $\{\Phi_{s,t}:0\leqslant s\leqslant t\leqslant T\}$ is a stochastic flow, i.e. $\Phi_{s,s}=\mathrm{Id}$ and*

$$\Phi_{s,t}\circ\Phi_{r,s}=\Phi_{r,t}\qquad \textit{for all } 0\leqslant r\leqslant s\leqslant t\leqslant T.$$

(b) *the weak solution $(Y(t):t\in[0,T])$ is a Markov process with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.*

Proof. (a). We first show that for all $0 \leqslant r \leqslant s \leqslant t \leqslant T$ we have

$$T(t-s)\left(\int_r^s T(s-q)B\,\mathrm{d}L(q)\right) = \int_r^s T(t-q)B\,\mathrm{d}L(q). \tag{5.0.15}$$

For any $v \in V$, we obtain by (2.4.2)

$$\begin{aligned}
\left\langle T(t-s)\left(\int_r^s T(s-q)B\,\mathrm{d}L(q)\right), v\right\rangle &= \left\langle \int_r^s T(s-q)B\,\mathrm{d}L(q), T^*(t-s)v\right\rangle \\
&= \int_r^s B^*T^*(s-q)(T^*(t-s)v)\,\mathrm{d}L(q) \\
&= \int_r^s B^*T^*(t-q)v\,\mathrm{d}L(q) \\
&= \left\langle \int_r^s T(t-q)B\,\mathrm{d}L(q), v\right\rangle,
\end{aligned}$$

which shows (5.0.15). This enables us to conclude

$$\begin{aligned}
\Phi_{s,t}(\Phi_{r,s}(v)) &= T(t-s)\Phi_{r,s}(v) + \int_s^t T(t-q)B\,\mathrm{d}L(q) \\
&= T(t-s)\left(T(s-r)v + \int_r^s T(s-q)B\,\mathrm{d}L(q)\right) + \int_s^t T(t-q)B\,\mathrm{d}L(q) \\
&= T(t-r)v + \int_r^s T(t-q)B\,\mathrm{d}L(q) + \int_s^t T(t-q)B\,\mathrm{d}L(q) \\
&= T(t-r)v + \int_r^t T(t-q)B\,\mathrm{d}L(q) \\
&= \Phi_{r,t}(v),
\end{aligned}$$

which completes the proof of (a).

(b) By the definition of stochastic integrals, we deduce that each $\Phi_{s,t}(v)$ is measurable with respect to $\sigma(\{L(p)u - L(q)u :\, s \leqslant q < p \leqslant t,\, u \in U\})$ for each $v \in V$. The independent increments of L guarantee that $\Phi_{s,t}(v)$ is independent of $\mathcal{F}_s$. Consequently, by using Part (a) we obtain for any bounded, measurable function $f\colon V \to \mathbb{R}$ that

$$E\big[f(\Phi_{0,t+s}(Y_0))|\mathcal{F}_s\big] = E\big[f(\Phi_{s,t+s} \circ \Phi_{0,s}(Y_0))|\mathcal{F}_s\big] = g_{s,t,f}(\Phi_{0,s}(Y_0)),$$

where $g_{s,t,f}(v) := E\big[f(\Phi_{s,t+s}(v))\big]$ for $v \in V$. Since $E\big[f(\Phi_{0,t+s}(Y_0))|\Phi_{0,s}(Y_0)\big] = g_{s,t,f}(\Phi_{0,s}(Y_0))$ we obtain

$$E\big[f(\Phi_{0,t+s}(Y_0))|\mathcal{F}_s\big] = E\big[f(\Phi_{0,t+s}(Y_0))|\Phi_{0,s}(Y_0)\big],$$

which completes the proof of Part (b). $\square$

Chapter 6

Stationarity

This chapter is based on some unpublished joint work with my supervisor.

6.1 Introduction

In this chapter, we study the necessary and sufficient conditions for the existence of an invariant measure (stationary measure) for the solution process of the stochastic Cauchy problem.

6.2 Invariant measure

Recall that in Chapter 4 we have proved that the stochastic Cauchy problem

$$\begin{aligned} \mathrm{d}Y(t) &= AY(t)\,\mathrm{d}t + B\,\mathrm{d}L(t) \qquad \text{for all } t \in [0,T], \\ Y(0) &= Y_0, \end{aligned} \tag{6.2.1}$$

has a unique (up to modification) weak solution in $[0,T]$ if and only if the map $s \to T(s)B$ is stochastically integrable with respect to L in $[0,T]$. In the Lemma below we show that if the solution exists in $[0,T]$, it can be extended to $\mathbb{R}_+$.

Lemma 6.2.1. If there exists a weak solution for the stochastic Cauchy problem (6.2.1) in $[0,T]$ for some $T>0$, then there exists a weak solution in $[0,\infty)$.

Proof. By Theorem 4.4.2 the map $s \to T(s)B$ is stochastically integrable in $[0,T]$. We show that the map $s \to T(s)B$ is stochastically integrable in $[0,S]$ for any $S>0$. Given $S>0$, choose $M \in \mathbb{N}$ such that $S/M \leqslant T$. Define the cylindrical random variable

$$Z\colon V \to L^0_P(\Omega;\mathbb{R}), \qquad Zv := \int_0^S B^*T^*(s)v\,\mathrm{d}L(s).$$

We show that Z is induced by a V-valued random variable. By [46, Lemma 5.4] and the semigroup property, we obtain that for each $v \in V$,

$$\begin{aligned}
\varphi_Z(v) &= \exp\left(\int_0^S \Psi(B^*T^*(s)v)\,\mathrm{d}s\right)\\
&= \exp\left(\sum_{i=0}^{M-1}\int_{\frac{iS}{M}}^{\frac{(i+1)S}{M}} \Psi(B^*T^*(s)v)\,\mathrm{d}s\right)\\
&= \prod_{i=0}^{M-1}\exp\left(\int_{\frac{iS}{M}}^{\frac{(i+1)S}{M}} \Psi(B^*T^*(s)v)\,\mathrm{d}s\right)\\
&= \prod_{i=0}^{M-1}\exp\left(\int_0^{\frac{S}{M}} \Psi\left(B^*T^*\left(s+\frac{iS}{M}\right)v\right)\,\mathrm{d}s\right)\\
&= \prod_{i=0}^{M-1}\exp\left(\int_0^{\frac{S}{M}} \Psi\left(B^*T^*(s)T^*\left(\frac{iS}{M}\right)v\right)\,\mathrm{d}s\right).
\end{aligned} \tag{6.2.2}$$

By stochastic integrability of the map $s \to T(s)B$ in $[0,T]$, there exists a V-valued random variable $I_{[0,S/M]}$ with distribution ϑ such that its characteristic function is given by

$$\varphi_\vartheta(v) = \exp\left(\int_0^{\frac{S}{M}} \Psi\left(B^*T^*(s)v\right)\,\mathrm{d}s\right). \tag{6.2.3}$$

If for each $i \in \{0,\dots,M-1\}$, the image measure $\vartheta \circ T\left(\frac{iS}{M}\right)^{-1}$ is denoted by λ_i and $\lambda :=$

$\lambda_0 * \cdots * \lambda_{M-1}$, then by (6.2.2) and (6.2.3) it follows that

$$\varphi_Z(v) = \varphi_\lambda(v) \qquad \text{for all } v \in V.$$

It implies by [53, Theorem IV.2.5] that Z is induced by a genuine V-valued random variable. Hence $s \to T(s)B$ is stochastically integrable in $[0, S]$ which completes the proof by Theorem 4.3.1. □

In the rest of this chapter we assume that the map $s \to T(s)B$ is stochastically integrable with respect to L in $[0, T]$ for some (and hence each) $T > 0$. The weak solution of (6.2.1) is given by the process $(Y(t) : t \geqslant 0)$ where

$$Y(t) = T(t)Y_0 + \int_0^t T(t-s)B \, \mathrm{d}L(s), \quad t \geqslant 0. \tag{6.2.4}$$

Recall from Section 4.2 that for each $t \geqslant 0$,

$$\int_0^t T(t-s)B \, \mathrm{d}L(s) \stackrel{d}{=} \int_0^t T(s)B \, \mathrm{d}L(s),$$

and if ν_t denotes the distribution of the random variable $\int_0^t T(s)B \, \mathrm{d}L(s)$, then ν_t is an infinitely divisible distribution with characteristics (c_t, S_t, ξ_t), given by

$$\langle c_t, v \rangle = \int_0^t a(B^*T^*(s)v) \, \mathrm{d}s + \int_V \langle h, v \rangle \left(\mathbb{1}_{B_V}(h) - \mathbb{1}_{B_\mathbb{R}}(\langle h, v \rangle) \right) \xi_t(\mathrm{d}h), \tag{6.2.5}$$

$$\langle v, S_t v \rangle = \int_0^t \langle B^*T^*(s)v, QB^*T^*(s)v \rangle \, \mathrm{d}s, \tag{6.2.6}$$

$$\xi_t = (\mathrm{leb} \otimes \mu) \circ \chi_t^{-1} \quad \text{on } \mathcal{Z}(V), \tag{6.2.7}$$

where $\chi_t \colon [0, \infty) \times U \to V$ is defined by $\chi_t(s, u) := \mathbb{1}_{[0,t]}(s) T(s) B u$.

Lemma 6.2.2. The process $\left(\int_0^t T(s)B\,\mathrm{d}L(s) : t \geqslant 0\right)$ has independent increments.

Proof. By construction of the stochastic integral, the random variable $\int_0^t T(s)B\,\mathrm{d}L(s)$ is $\mathcal{F}_t$-measurable for each $t \geqslant 0$. It is enough to show that $\int_s^t T(r)B\,\mathrm{d}L(r)$ is independent of $\mathcal{F}_s$ for $s < t$. If $g\colon [s,t] \to U$ is a regulated function, then there exists a sequence of simple functions g_n such that $g_n \to g$ in $R([s,t];U)$. Suppose for each n, g_n takes values $u_{n,k}$ on the interval $(t_k^{(n)}, t_{k+1}^{(n)})$ for $k = 0, 1, ..., N_n - 1$, where $\{t_k^n\}_{k=0}^{N_n-1}$ is a partition of $[s,t]$. By definition of the stochastic integral, we have

$$\int_s^t g_n(r)\,\mathrm{d}L(r) = \sum_{k=0}^{N_n-1} \left(L(t_{k+1}^{(n)}) - L(t_k^{(n)})\right)(u_{n,k}).$$

Then for each $k = 0, 1, ..., N_n - 1$, the increment $\left(L\left(t_{k+1}^{(n)}\right) - L\left(t_k^{(n)}\right)\right)(u_{n,k})$ is measurable with respect to the σ-algebra

$$\sigma\left(\{(L(q) - L(p))u : s \leqslant p \leqslant q \leqslant t, u \in U\}\right),$$

which is independent of $\mathcal{F}_s$. Consequently, $\lim_{n\to\infty} \int_s^t g_n(r)\,\mathrm{d}L(r) = \int_s^t g(r)\,\mathrm{d}L(r)$ is also independent of $\mathcal{F}_s$. By the stochastic integrability of the map $r \to T(r)B$, we have for each $v \in V$,

$$\left\langle v, \int_s^t T(r)B\,\mathrm{d}L(r) \right\rangle = \int_s^t B^*T^*(r)v\,\mathrm{d}L(r). \tag{6.2.8}$$

Since the map $r \mapsto g(r) := B^*T^*(r)v$ is a regulated function, the right side in (6.2.8) is independent of $\mathcal{F}_s$ resulting in the independence of $\int_s^t T(r)B\,\mathrm{d}L(r)$ and $\mathcal{F}_s$. $\square$

For notational convenience, we will denote the image measure $\nu \circ T(t)^{-1}$ of a measure ν

on $\mathfrak{B}(V)$ by $T_t\nu$, that is,

$$T_t\nu(B) = \nu(T(t)^{-1}(B)) \qquad \text{for all } B \in \mathfrak{B}(V).$$

Lemma 6.2.3. The family $\{\nu_t : t \geqslant 0\}$ of probability measures in $\mathfrak{B}(V)$ satisfy

$$\nu_{t+s} = T_t\nu_s * \nu_t \qquad \text{for all } s, t \geqslant 0. \tag{6.2.9}$$

Proof. Let $\varphi_{T_t\nu_s*\nu_t} \colon V \to \mathbb{C}$ denote the characteristic function of the probability measure $T_t\nu_s * \nu_t$. For each $v \in V$ and $s, t \geqslant 0$, we obtain,

$$\begin{aligned}
\varphi_{T_t\nu_s*\nu_t}(v) &= \varphi_{T_t\nu_s}(v)\varphi_{\nu_t}(v) \\
&= \varphi_{\nu_s}(T^*(t)v)\varphi_{\nu_t}(v) \\
&= \exp\left(\int_0^s \Psi\left(B^*T^*(r)T^*(t)v\right)\,\mathrm{d}s\right)\exp\left(\int_0^t \Psi\left(B^*T^*(r)v\right)\,\mathrm{d}s\right) \\
&= \exp\left(\int_0^s \Psi\left(B^*T^*(r+t)v\right)\,\mathrm{d}s + \int_0^t \Psi\left(B^*T^*(r)v\right)\,\mathrm{d}s\right) \\
&= \exp\left(\int_t^{t+s} \Psi\left(B^*T^*(r)v\right)\,\mathrm{d}s + \int_0^t \Psi\left(B^*T^*(r)v\right)\,\mathrm{d}s\right) \\
&= \exp\left(\int_0^{t+s} \Psi\left(B^*T^*(r)v\right)\,\mathrm{d}s\right) \\
&= \varphi_{\nu_{t+s}}(v),
\end{aligned}$$

which proves that (6.2.9) is satisfied. $\square$

A family of measures satisfying (6.2.9) is called *skew-convolution* semigroup or $(T(t))$-convolution semigroup. For any function $\Phi \colon [0, \infty) \to \mathcal{L}(U, V)$ which is stochastically integrable in $[0, T]$ for each $T > 0$, we define the integral $\int_0^\infty \Phi(s)\,\mathrm{d}L(s)$ as the limit in probability (if it exists) as $t \to \infty$ of the integrals $\int_0^t \Phi(s)\,\mathrm{d}L(s)$.

Lemma 6.2.4. The following conditions are equivalent:

(i) $\{\nu_t\}$ is weakly convergent as $t \uparrow \infty$;

(ii) $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ exists.

In this case, $\lim_{t\to\infty} \nu_t = \mathscr{L}\left(\int_0^\infty T(s)B\,\mathrm{d}L(s)\right)$.

Proof. For each $t \geqslant 0$, we have

$$\int_0^t T(t-s)B\,\mathrm{d}L(s) \overset{d}{=} \int_0^t T(r)B\,\mathrm{d}L(r).$$

By Lemma 6.2.2, $\left(\int_0^t T(r)B\,\mathrm{d}L(r), t \geqslant 0\right)$ is a process with independent increments, and as a consequence of [29, Lemma A.2.1] it converges as $t \to \infty$ in distribution if and only if it converges in probability . Hence ν_t is weakly convergent as $t \uparrow \infty$ (i.e., $\int_0^t T(t-s)B\,\mathrm{d}L(s)$ converges in distribution) if and only if $\int_0^t T(u)B\,\mathrm{d}L(u)$ converges in probability as $t \uparrow \infty$ (i.e., $\int_0^\infty T(s)\,\mathrm{d}L(s)$ exists). □

Definition 6.2.5. A probability measure ν on $\mathfrak{B}(V)$ is called a *stationary measure* for the process $(Y(t), t \geqslant 0)$ defined in (6.2.4) if it satisfies

$$\nu = T_t\nu * \nu_t \qquad \text{for all } t \geqslant 0. \tag{6.2.10}$$

In the literature, any measure satisfying (6.2.10) is also called an *operator self-decomposable measure*. The concept of an operator self-decomposable measure or an operator self-decomposable random variable in Banach spaces was first introduced by Urbanik in [52]. This concept is also studied among others by Jurek [28], Jurek and Vervaat [29], Sato and Yamazato [48] [49], Applebaum [2], [3], [5]. A stationary measure can also be defined as the invariant measure for the generalised Mehler semigroup (or transition semigroup) of the process Y. We now discuss this approach and show the equivalence of the two approaches. The concept of a generalised Mehler semigroup has been studied in detail in [10] for the Gaussian case and [24] for the non-Gaussian case. The generalised Mehler semigroup $(P_t : t \geqslant 0)$ for the family $\{\nu_t : t \geqslant 0\}$ is defined by the formula

$$P_t \colon B_b(V) \to B_b(V), \qquad P_t f(v) = \int_V f(T(t)v + h)\nu_t(\mathrm{d}v),$$

for any $f \in B_b(V)$, where $B_b(V)$ denotes the space of all bounded and Borel measurable functions on V. The generalised Mehler semigroup is a semigroup by [10, Prop. 2.2] because $\{\nu_t : t \geqslant 0\}$ is a skew-convolution semigroup by Lemma 6.2.3. A measure ν is called an invariant measure for the transition semigroup $(P_t : t \geqslant 0)$ if for all $f \in B_b(V)$ and $t \geqslant 0$,

$$\int_V P_t f(v)\nu(\mathrm{d}v) = \int_V f(v)\nu(\mathrm{d}v). \tag{6.2.11}$$

The following equivalence result is from [3, Theorem 2.1], whose proof we give for the sake of completeness.

Theorem 6.2.6. *The following are equivalent for a measure ν on $\mathfrak{B}(V)$:*

(a) ν is a stationary measure for the process (6.2.4)*, i.e., it satisfies* (6.2.10)*;*

(b) ν is an invariant measure for the generalised Mehler semigroup $(P_t : t \geqslant 0)$.

(c) The process $(Y(t) : t \geqslant 0)$ is a strictly stationary process with ν being the distribution of Y_0.

Proof. (a) $\Rightarrow$ (b). For any $f \in B_b(V)$ and $t \geqslant 0$, Fubini's theorem implies that

$$\int_V f(v)\nu(\mathrm{d}v) = \int_V f(v)\,(T_t\nu * \nu_t)(\mathrm{d}v)$$

$$
\begin{aligned}
&= \int_V \int_V f(v+h)\,(T_t\nu)(\mathrm{d}v)\,\nu_t(\mathrm{d}h) \\
&= \int_V \int_V f(T(t)v+h)\,\nu(\mathrm{d}v)\,\nu_t(\mathrm{d}h) \\
&= \int_V P_t f(v)\,\nu(\mathrm{d}v).
\end{aligned}
$$

(b) $\Rightarrow$ (c). This is a standard result e.g. see [20, Prop. 11.5].

(c) $\Rightarrow$ (a). If the process Y is a strictly stationary process, then for each $t \geqslant 0$, we have $Y(0) \stackrel{d}{=} Y(t)$ which implies

$$
Y_0 \stackrel{d}{=} Y(t) = T(t)Y_0 + \int_0^t T(t-s)\,\mathrm{d}L(s).
$$

If ν is the distribution of Y_0, then ν satisfies (6.2.10) and hence it is a stationary measure for the process (6.2.4). □

Lemma 6.2.7. If $\{\nu_t\}$ converges weakly to ν in $\mathcal{M}(V)$ as $t \to \infty$, then

(i) ν is an invariant measure for the process (6.2.4);

(ii) any invariant measure λ of (6.2.4) has the form $\lambda = \beta * \nu$, where β is a probability measure such that $\beta = T_t\beta$ for all $t \geqslant 0$.

Proof. By Lemma 6.2.3, for any $s, t \geqslant 0$, we have

$$
\nu_{t+s} = T_t\nu_s * \nu_t.
$$

Taking limit as $s \to \infty$, we obtain

$$
\nu = T_t\nu * \nu_t \qquad \text{for all } t \geqslant 0,
$$

which proves the first part. To prove the second part, we follow the arguments in [16, Prop.

3.2]. Let λ be an invariant measure for (6.2.4) and $t_n \to \infty$. By the definition of the invariant measure,

$$\lambda = T_{t_n}\lambda * \nu_{t_n} \qquad \text{for all } n \in \mathbb{N}\,. \tag{6.2.12}$$

By assumption $\{\nu_{t_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{M}(V)$, and $\{\lambda\}$ is trivially relatively compact, therefore, by [36, Theorem III.2.1, p. 58], the sequence $\{T_{t_n}\lambda : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{M}(V)$. As a consequence of infinite divisibility of distributions ν and ν_t, we obtain $\varphi_\nu(v) \neq 0$, $\varphi_{\nu_t}(v) \neq 0$ for all $v \in V$. It follows by (6.2.12) that,

$$\varphi_{T_{t_n}\lambda}(v) = \frac{\varphi_\lambda(v)}{\varphi_{\nu_{t_n}}(v)} \to \frac{\varphi_\lambda(v)}{\varphi_\nu(v)} \qquad \text{as } n \to \infty.$$

Hence by [36, Lemma VI.2.1] and the fact that $(t_n)_{n\in\mathbb{N}}$ is an arbitrary sequence, $\{T_t\lambda\}$ converges weakly to some probability measure β and $\lambda = \beta * \nu$. Using that both λ and ν are stationary measures for (6.2.4), we have,

$$\beta * \nu = \lambda = T_t\lambda * \nu_t = T_t(\beta * \nu) * \nu_t = T_t\beta * (T_t\nu * \nu_t) = T_t\beta * \nu.$$

Consequently, $\varphi_\beta(v)\varphi_\nu(v) = \varphi_{T_t\beta}(v)\varphi_\nu(v)$ for all $v \in V$. Since $\varphi_\nu(v) \neq 0$, for all $v \in V$, we get $\varphi_\beta(v) = \varphi_{T_t\beta}(v)$ implying $\beta = T_t\beta$.

□

By Lemma 6.2.4 and 6.2.7, if $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ exists then its law is an invariant measure. Thus, conditions for the existence of the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ give us the conditions for the existence of an invariant measure. We will see in the next section that in the case of stable semigroups, the existence of invariant measure also implies that $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ exists implying thereby that the invariant measure, if exists, is unique.

The next theorem is a generalisation of the results in [16] for cylindrical Lévy processes.

Theorem 6.2.8. *The following conditions are necessary and sufficient for the existence of the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$.*

(a) There exists

$$c_\infty := \lim_{t\to\infty} c_t \qquad \textit{in } V; \tag{6.2.13}$$

(b) $$\int_0^\infty \operatorname{tr}\big[T(s)BQB^*T^*(s)\big]\,\mathrm{d}s < \infty; \tag{6.2.14}$$

(c) $$\sup_{n\geqslant 1}\int_0^\infty \int_U \left(\sum_{k=1}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s < \infty; \tag{6.2.15}$$

(d) $$\limsup_{m\to\infty}\sup_{n\geqslant m}\int_0^\infty \int_U \left(\sum_{k=m}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s = 0. \tag{6.2.16}$$

We need the following Lemmas to prove this theorem.

Lemma 6.2.9. If (6.2.15) holds, then the cylindrical measure $\eta := (\mathrm{leb}\otimes\mu)\circ\chi_{[0,\infty)}^{-1}$ is a cylindrical Lévy measure in V, where $\chi_{[0,\infty)}\colon [0,\infty)\times U \to V$ is defined by $\chi_{[0,\infty)}(s,u) := T(s)Bu$.

Proof. From (6.2.15), it follows that

$$\begin{aligned}
&\sup_{n\geqslant 1}\int_{\mathbb{R}^n} \left(|\beta|^2\wedge 1\right)\eta\circ\pi_{h_1,\dots,h_n}^{-1}(\mathrm{d}\beta)\\
&\qquad = \sup_{n\geqslant 1}\int_0^\infty \int_U \left(\sum_{k=1}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s\\
&\qquad < \infty.
\end{aligned} \tag{6.2.17}$$

By defining the map $\pi_n\colon V \to V$ by $\pi_n(v) := \sum_{k=1}^n \langle v, h_k\rangle h_k$, we have for any $v \in V$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\int_0^\infty \int_U \Big(\langle T(s)Bu, \pi_n(v)\rangle^2 \wedge 1\Big)\,\mu(\mathrm{d}u)\,\mathrm{d}s \\
&\qquad = \int_V \Big(\langle h, \pi_n(v)\rangle^2 \wedge 1\Big)\,\eta(\mathrm{d}h) \\
&\qquad = \int_{\mathbb{R}^n} \Big(\langle \beta, \pi_{h_1,\dots,h_n}(v)\rangle^2 \wedge 1\Big)\Big(\eta \circ \pi^{-1}_{h_1,\dots,h_n}\Big)(\mathrm{d}\beta) \\
&\qquad \leqslant \max\{1, \|v\|^2\} \int_{\mathbb{R}^n} (|\beta|^2 \wedge 1)\Big(\eta \circ \pi^{-1}_{h_1,\dots,h_n}\Big)(\mathrm{d}\beta) \qquad (6.2.18)
\end{aligned}$$

From (6.2.17) and (6.2.18), we conclude that

$$\sup_{n\geqslant 1} \int_0^\infty \int_U \Big(\langle T(s)Bu, \pi_n(v)\rangle^2 \wedge 1\Big)\,\mu(\mathrm{d}u)\,\mathrm{d}s < \infty. \qquad (6.2.19)$$

Since for any sequence $(u_n)_{n\in\mathbb{N}} \subset U$ satisfying $u_n \to u$ in U, the finite measures $(|\beta|^2 \wedge 1)\mu \circ \langle \cdot, u_n\rangle^{-1}$ converge weakly to $(|\beta|^2 \wedge 1)\mu \circ \langle \cdot, u\rangle^{-1}$ by [44, Lemma 4.4], it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \int_U \Big(\langle T(s)Bu, \pi_n(v)\rangle^2 \wedge 1\Big)\,\mu(\mathrm{d}u) \\
&\qquad = \lim_{n\to\infty} \int_{\mathbb{R}} \big(|\beta|^2 \wedge 1\big)\Big(\mu \circ \langle \cdot, B^*T^*(s)\pi_n(v)\rangle^{-1}\Big)(\mathrm{d}\beta) \\
&\qquad = \int_{\mathbb{R}} \big(|\beta|^2 \wedge 1\big)\Big(\mu \circ \langle \cdot, B^*T^*(s)v\rangle^{-1}\Big)(\mathrm{d}\beta) \\
&\qquad = \int_U \Big(\langle T(s)Bu, v\rangle^2 \wedge 1\Big)\,\mu(\mathrm{d}u). \qquad (6.2.20)
\end{aligned}$$

By (6.2.19), (6.2.20) and Fatou's lemma, we have for each $v \in V$,

$$\begin{aligned}
\int_V \Big(\langle h, v\rangle^2 \wedge 1\Big)\,\eta(\mathrm{d}h) &= \int_0^\infty \int_U \Big(\langle T(s)Bu, v\rangle^2 \wedge 1\Big)\,\mu(\mathrm{d}u)\,\mathrm{d}s \\
&\leqslant \liminf_n \int_0^\infty \int_U \Big(\langle T(s)Bu, \pi_n(v)\rangle^2 \wedge 1\Big)\,\mu(\mathrm{d}u)\,\mathrm{d}s
\end{aligned}$$

$$< \infty,$$

which proves that η is a cylindrical Lévy measure. $\square$

Lemma 6.2.10. If (6.2.15) holds, then the mapping

$$f\colon V \to \mathbb{C}, \qquad f(v) := \int_V \left(\cos(\langle h, v\rangle) - 1\right) \eta(\mathrm{d}h)$$

satisfies $f(\pi_n v) \to f(v)$ as $n \to \infty$ for each $v \in V$.

Proof. (The proof is based on some arguments used in the proof of [46, Lemma 5.1].) We first note that by monotone convergence theorem and (6.2.15), it follows that

$$\begin{aligned}
&\int_0^\infty \sup_{n\geqslant 1} \int_U \left(\sum_{k=1}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\quad = \sup_{n\geqslant 1} \int_0^\infty \int_U \left(\sum_{k=1}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s < \infty.
\end{aligned} \tag{6.2.21}$$

Let $v \in V$ be fixed. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
f(\pi_n v) &= \int_0^\infty \int_U \left(\cos(\langle T(s)Bu, \pi_n v\rangle) - 1\right) \mu(\mathrm{d}u)\,\mathrm{d}s, \\
&=: \int_0^\infty \alpha_n(s)\,\mathrm{d}s,
\end{aligned}$$

where the mapping α_n is given by

$$\alpha_n\colon [0,\infty) \to \mathbb{C}, \qquad \alpha_n(s) := \int_U \left(\cos\left(\langle T(s)Bu, \pi_n v\rangle\right) - 1\right) \mu(\mathrm{d}u).$$

Define the function

$$g\colon \mathbb{R} \to \mathbb{C}, \qquad g(\beta) = \begin{cases} \frac{\cos(\beta)-1}{\beta^2\wedge 1}, & \text{if } \beta \neq 0, \\ -\frac{1}{2}, & \text{if } \beta = 0. \end{cases}$$

It follows easily that g is bounded and continuous. By using Lemma 4.4 in [44], we have

$$\begin{aligned}
\lim_{n\to\infty} \alpha_{v_n}(s) &= \lim_{n\to\infty} \int_U g\left(\langle u, B^*T^*(s)\pi_n v\rangle\right) \left(\langle u, B^*T^*(s)\pi_n v\rangle^2 \wedge 1\right) \mu(\mathrm{d}u) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} g(\beta) \left(|\beta|^2 \wedge 1\right) \left(\mu \circ \langle \cdot, B^*T^*(s)\pi_n v\rangle^{-1}\right)(\mathrm{d}\beta) \\
&= \int_{\mathbb{R}} g(\beta) \left(|\beta|^2 \wedge 1\right) \left(\mu \circ \langle \cdot, B^*T^*(s) v\rangle^{-1}\right)(\mathrm{d}\beta) \\
&=: \alpha(s)
\end{aligned} \tag{6.2.22}$$

For each $n \in \mathbb{N}$ and $s \geqslant 0$, we have

$$\begin{aligned}
|\alpha_{v_n}(s)| &= \left| \int_U g\left(\langle u, B^*T^*(s)\pi_n v\rangle\right) \left(\langle u, B^*T^*(s)\pi_n v\rangle^2 \wedge 1\right) \mu(\mathrm{d}u) \right| \\
&\leqslant \|g\|_\infty \int_U \left(\langle T(s)Bu, \pi_n v\rangle^2 \wedge 1\right) \mu(\mathrm{d}u) \\
&= \|g\|_\infty \int_U \left(\langle \pi_n T(s)Bu, v\rangle^2 \wedge 1\right) \mu(\mathrm{d}u) \\
&\leqslant \|g\|_\infty \max\{1, \|v\|^2\} \int_U \left(\|\pi_n T(s)Bu\|^2 \wedge 1\right) \mu(\mathrm{d}u) \\
&= \|g\|_\infty \max\{1, \|v\|^2\} \int_U \left(\sum_{k=1}^n \langle T(s)Bu, h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u) \\
&\leqslant \|g\|_\infty \max\{1, \|v\|^2\} \sup_{n\geqslant 1} \int_U \left(\sum_{k=1}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)
\end{aligned} \tag{6.2.23}$$

In view of (6.2.21), (6.2.22) and (6.2.23), Lebesgue's theorem on dominated convergence implies that

$$\lim_{n\to\infty}\int_0^\infty \alpha_n(s)\,\mathrm{d}s = \int_0^\infty \alpha(s)\,\mathrm{d}s,$$

which completes the proof. $\square$

Lemma 6.2.11. The following conditions are equivalent:

(a) For each Borel set $A \in \mathfrak{B}([0,\infty)$, the cylindrical measure $(\mathrm{leb}\otimes\mu)\circ\chi_A^{-1}$ extends to a Lévy measure on $\mathfrak{B}(V)$, where $\chi_A\colon [0,\infty)\times U \to V$ is defined by $\chi_A(s,u) := \mathbb{1}_A(s)T(s)Bu$.

(b) Conditions (6.2.15) and (6.2.16) are satisfied.

Proof. (a) $\Rightarrow$ (b). The result follows easily by noting that $(\mathrm{leb}\otimes\mu)\circ\chi_{[0,\infty)}^{-1}$ extends to a Lévy measure, and making use of the definition of Lévy measure, monotone convergence theorem and Lebesgue's theorem.

(b) $\Rightarrow$ (a). For any $N\in\mathbb{N}$, let $\rho_N := (\eta+\eta^-)\circ\pi_N^{-1}$, where $\eta^-(C) := \eta(-C)$ for all $C\in\mathcal{Z}(V)$ and $\pi_N\colon V\to V$ is defined by $\pi_N(v) = \sum_{k=1}^N \langle v,h_k\rangle\, h_k$. Then for each $N\in\mathbb{N}$, by using (6.2.15) we obtain

$$\begin{aligned}\int_V \left(\|v\|^2\wedge 1\right)\rho_N(\mathrm{d}v) &= \int_V \left(\|\pi_N v\|^2\wedge 1\right)(\eta+\eta^-)(\mathrm{d}v)\\ &= 2\int_V \left(\sum_{k=1}^N \langle v,h_k\rangle^2\wedge 1\right)\eta(\mathrm{d}v)\\ &= 2\int_0^\infty\int_U \left(\sum_{k=1}^N \langle u, B^*T^*(s)h_k\rangle^2\wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s\\ &< \infty,\end{aligned}$$

which implies that ρ_N is a Lévy measure in $\mathfrak{B}(V)$. By Lévy-Khinchine Theorem, there exists an infinitely divisible probability measure θ_N with characteristics $(0,0,\rho_N)$ such that its characteristic function is given by

$$\varphi_{\theta_N}\colon V \to \mathbb{C}, \qquad \varphi_{\theta_N}(v) := \exp\left(\int_V \left(\cos\left(\langle v,h\rangle\right) - 1\right)\rho_N(\mathrm{d}h)\right)$$

We follow the same arguments as in the proof of Theorem 5.10 in [46]. By an application of the inequality $1-\cos\beta \leqslant 2(\beta^2 \wedge 1)$ for all $\beta \in \mathbb{R}$, it follows that for every $v \in V$

$$\begin{aligned}
1-\varphi_{\theta_N}(v) &= 1-\exp\left(\int_V \cos(\langle v,h\rangle - 1)\rho_N(\mathrm{d}h)\right)\\
&\leqslant \int_V (1-\cos(\langle v,h\rangle)\rho_N(\mathrm{d}h)\\
&\leqslant 2\int_V (\langle h,v\rangle^2 \wedge 1)\rho_N(\mathrm{d}h).
\end{aligned}$$

By g_m, we denote the density of the standard normal distribution on $\mathfrak{B}(\mathbb{R}^m)$. For every $m,n \in \mathbb{N}$ with $m \leqslant n$ and $N \in \mathbb{N}$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^{n-m+1}} \left(1-\mathrm{Re}\varphi_{\theta_N}(\beta_m h_m + \cdots + \beta_n h_n)\right) g_{n-m+1}(\beta_m,\ldots,\beta_n)\,\mathrm{d}\beta_m \cdots \mathrm{d}\beta_n\\
&\quad\leqslant 2\int_{\mathbb{R}^{n-m+1}}\int_V \left(\left|\sum_{k=m}^n \beta_k \langle h,h_k\rangle\right|^2 \wedge 1\right)\rho_N(\mathrm{d}h) g_{n-m+1}(\beta_m,\ldots,\beta_n)\,\mathrm{d}\beta_m\cdots\mathrm{d}\beta_n\\
&\quad\leqslant 2\int_V \left(\left(\int_{\mathbb{R}^{n-m+1}}\left|\sum_{k=m}^n \beta_k\langle h,h_k\rangle\right|^2 g_{n-m+1}(\beta_m,\ldots,\beta_n)\,\mathrm{d}\beta_m\cdots\mathrm{d}\beta_n\right)\wedge 1\right)\rho_N(\mathrm{d}h)\\
&\quad= 2\int_V \left(\sum_{k=m}^n \langle h,h_k\rangle^2 \wedge 1\right)\rho_N(\mathrm{d}h)\\
&\quad= 2\int_V \left(\sum_{k=m}^n \langle \pi_N h,h_k\rangle^2 \wedge 1\right)(\eta+\eta^-)(\mathrm{d}v)\\
&\quad\leqslant 4\int_V \left(\sum_{k=m}^n \langle h,h_k\rangle^2 \wedge 1\right)\eta(\mathrm{d}h)\\
&\quad= 4\int_0^\infty \int_U \left(\sum_{k=m}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s.
\end{aligned}$$

Condition (6.2.16) implies

$$\limsup_{m\to\infty} \sup_{n\geqslant m} \sup_{N\in\mathbb{N}} \int_{\mathbb{R}^{n-m+1}} (1 - \mathrm{Re}\varphi_{\theta_N}(\beta_m h_m + \cdots + \beta_n h_n))\, g_{n-m+1}(\beta_m, \ldots, \beta_n)\, \mathrm{d}\beta_m \cdots \mathrm{d}\beta_n = 0.$$

Let $n \in \mathbb{N}$ be fixed. For each $N \in \mathbb{N}$, define the function

$$\psi_N \colon \mathbb{R}^n \to \mathbb{C}, \qquad \psi_N(\beta) := \varphi_{\theta_N}(\beta_1 h_1 + \cdots + \beta_n h_n) \text{ for all } \beta = (\beta_1, \ldots, \beta_n) \text{ in } \mathbb{R}^n.$$

We show that the family $(\psi_N : N \in \mathbb{N})$ is equicontinuous at the origin. We have

$$\psi_N(\beta) = \exp\left(\int_V (\cos(\langle \beta_1 h_1 + \cdots + \beta_n h_n, h\rangle) - 1)\, \rho_N(\mathrm{d}h)\right) \tag{6.2.24}$$

Using (6.2.17) and Lebesgue's theorem, we obatin

$$\begin{aligned}
&\left|\int_V (\cos(\langle \beta_1 h_1 + \cdots + \beta_n h_n, h\rangle) - 1)\, \rho_N(\mathrm{d}h)\right| \\
&\qquad \leqslant \int_V |\cos(\langle \beta_1 h_1 + \cdots + \beta_n h_n, h\rangle) - 1|\, \rho_N(\mathrm{d}h) \\
&\qquad \leqslant 2\int_V \left(\langle \beta_1 h_1 + \cdots + \beta_n h_n, h\rangle^2 \wedge 1\right) \rho_N(\mathrm{d}h) \\
&\qquad = 4\int_V \left(\langle \beta_1 h_1 + \cdots + \beta_n h_n, \pi_N h\rangle^2 \wedge 1\right) \eta(\mathrm{d}h) \\
&\qquad \leqslant 4\int_V \left(|\beta|^2 \sum_{k=1}^n \langle h_k, h\rangle^2 \wedge 1\right) \eta(\mathrm{d}h) \\
&\qquad = 4\int_{\mathbb{R}^n} (|\beta|^2 |\alpha|^2 \wedge 1) \left(\eta \circ \pi^{-1}_{h_1,\ldots,h_n}\right)(\mathrm{d}\alpha) \\
&\qquad \to 0 \quad \text{as } |\beta| \to 0,
\end{aligned}$$

where it is also clear that the convergence is uniform in N. This together with (6.2.24) implies that the family $(\psi_N : N \in \mathbb{N})$ is equicontinuous at the origin. It implies by [36, Lemma VI.2.3] that the family $\{\theta_N : N \in \mathbb{N}\}$ is relatively compact in $\mathcal{M}(V)$. Using Lemma 6.2.10, we obtain for each $v \in V$,

$$
\begin{aligned}
\lim_{N\to\infty} \varphi_{\theta_N}(v) &= \lim_{N\to\infty} \exp\left(\int_V \left(\cos\left(\langle v, h\rangle\right) - 1\right) \rho_N(\mathrm{d}h)\right) \\
&= \lim_{N\to\infty} \exp\left(\int_V \left(\cos\left(\langle \pi_N v, h\rangle\right) - 1\right) (\eta + \eta^-)(\mathrm{d}h)\right) \\
&= \exp\left(\int_V \left(\cos\left(\langle v, h\rangle\right) - 1\right) (\eta + \eta^-)(\mathrm{d}h)\right).
\end{aligned}
$$

It follows by [36, Lemma VI.2.1] that $\{\theta_n\}_{n\in\mathbb{N}}$ converges weakly to an infinitely divisible probability measure θ and the characteristic function of θ satisfies

$$
\varphi_\theta(v) = \exp\left(\int_V \left(\cos\left(\langle v, h\rangle\right) - 1\right) (\eta + \eta^-)(\mathrm{d}h)\right).
$$

Consequently, $\eta + \eta^-$ extends to the Lévy measure of θ. Since

$$
\eta(C) \leqslant \eta(C) + \eta^-(C) \qquad \text{for all } C \in \mathcal{Z}(V),
$$

Theorem 3.4 in [46] implies that η extends to a Lévy measure on $\mathfrak{B}(V)$. Finally for any Borel set $A \in [0, \infty)$, we have

$$
\begin{aligned}
\left((\mathrm{leb} \otimes \mu) \circ \chi_A^{-1}\right)(C) &= \int_A \int_U \mathbb{1}_C(T(s)Bu)\mu(\mathrm{d}u)\,\mathrm{d}s \\
&\leqslant \int_0^\infty \int_U \mathbb{1}_C(T(s)Bu)\mu(\mathrm{d}u)\,\mathrm{d}s = \eta(C) \qquad \text{for all } C \in \mathcal{Z}(V),
\end{aligned}
$$

from which Condition (a) follows by another application of [46, Theorem 3.4]. □

Proof of Theorem 6.2.8. Sufficiency: suppose that (6.2.13) -(6.2.16) hold. We first show that the family $\{\nu_t : t \geqslant 0\}$ is relatively compact in $\mathcal{M}(V)$, for which we use the compactness criterion for infinitely divisible probability measures as stated in Theorem 2.3.3. In view of (6.2.13), we only need to show that the set $(\xi_t : t \geqslant 0)$ restricted to the complement of any neighbourhood of the origin is relatively compact and the operators $R_t \colon V \to V$ defined by

$$\langle R_t v, v\rangle := \langle S_t v, v\rangle + \int_{\|h\|\leqslant 1} \langle v, h\rangle^2 \xi_t(\mathrm{d}h) \tag{6.2.25}$$

satisfy

(i) $\sup_{t\geqslant 0} \sum_{k=1}^{\infty} \langle R_t h_k, h_k\rangle < \infty$ and

(ii) $\lim_{N\to\infty} \sup_{t\geqslant 0} \sum_{k=N}^{\infty} \langle R_t h_k, h_k\rangle = 0.$

For any cylindrical set $C \in \mathcal{Z}(V)$, we have

$$\begin{aligned} \xi_t(C) &= \int_0^t \int_U \mathbb{1}_C(T(s)Bu)\mu(\mathrm{d}u)\,\mathrm{d}s \\ &\leqslant \int_0^\infty \int_U \mathbb{1}_C(T(s)Bu)\mu(\mathrm{d}u)\,\mathrm{d}s = (\mathrm{leb}\otimes\mu)\circ\chi_{[0,\infty)}^{-1}(C). \end{aligned} \tag{6.2.26}$$

It follows from Lemma 6.2.11 that the cylindrical measure $(\mathrm{leb}\otimes\mu)\circ\chi_{[0,\infty)}^{-1}$ extends to a Lévy measure on $\mathfrak{B}(V)$ which we denote by ξ_∞. Since $\mathfrak{B}(V)$ is the sigma algebra generated by $\mathcal{Z}(V)$ and $\mathcal{Z}(V)$ is a π-system, we obtain $\xi_t \leqslant \xi_\infty$ on $\mathfrak{B}(V)$ for all $t \geqslant 0$. Let ξ_t^c and ξ_∞^c denote the restrictions of the measures ξ_t^c and ξ_∞^c to the complement of any neighbourhood $V_1 \subset V$ of origin. By [33, Prop 1.1.3]) the finite measure ξ_∞^c is a Radon measure and therefore, for each $\varepsilon > 0$ there exists a compact set $K \subset V_1$ such that $\xi_\infty^c(K^c) \leqslant \varepsilon$. As a consequence,

$$\xi_t^c(K^c) \leqslant \xi_\infty^c(K^c) \leqslant \varepsilon \quad \text{for all } t \geqslant 0, \tag{6.2.27}$$

which implies that $\{\xi_t : t \geqslant 0\}$ restricted to the complement of any neighbourhood of the origin is relatively compact. We show that the operators $\{R_t\}$ in (6.2.25) satisfy (ii). Condition (i) can be proved analogously using (6.2.15). By Lebesgue's theorem on dominated convergence and (6.2.14), we have

$$\begin{aligned}\lim_{N\to\infty}\sup_{t\geqslant 0}\int_0^t \sum_{k=N}^{\infty}\langle T(s)BQB^*T^*(s)h_k, h_k\rangle \,\mathrm{d}s \\ = \lim_{N\to\infty}\int_0^\infty \sum_{k=N}^{\infty}\langle T(s)BQB^*T^*(s)h_k, h_k\rangle \,\mathrm{d}s = 0. \qquad (6.2.28)\end{aligned}$$

Condition (6.2.16) implies that

$$\begin{aligned}\sup_{t\geqslant 0}\sum_{k=N}^{\infty}\int_{\|h\|\leqslant 1}\langle h, h_k\rangle^2 \xi_t(\mathrm{d}h) &\leqslant \sup_{t\geqslant 0}\sup_{m\geqslant N}\int_V \left(\sum_{k=N}^{m}\langle h, h_k\rangle^2 \wedge 1\right)\xi_t(\mathrm{d}h) \\ &= \sup_{t\geqslant 0}\sup_{m\geqslant N}\int_0^t\int_U \left(\sum_{k=N}^{m}\langle T(s)Bu, h_k\rangle^2 \wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s \\ &= \sup_{m\geqslant N}\int_0^\infty\int_U \left(\sum_{k=N}^{m}\langle T(s)Bu, h_k\rangle^2 \wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s \\ &\to 0 \qquad \text{as } N\to\infty. \qquad (6.2.29)\end{aligned}$$

From (6.2.28) and (6.2.29), it follows that Condition (ii) is satisfied. Consequently $\{\nu_t : t \geqslant 0\}$ is relatively compact. Since $\{S_t\}$ is an increasing sequence of operators, Condition (6.2.14) implies that the operator

$$S_\infty := \int_0^\infty T(s)BQB^*T(s)\,\mathrm{d}s,$$

is well-defined and

$$\langle S_t v, v\rangle \to \langle S_\infty v, v\rangle \qquad \text{for all } v\in V. \qquad (6.2.30)$$

By a similar reasoning as in (6.2.26), we obtain that $\{\xi_t\}$ is an increasing family of Lévy measures and $\xi_t(A)$ increases to $\xi_\infty(A)$ for each $A \in \mathfrak{B}(V)$. If for a fixed $v \in V$, we define

$$K(h,v) := e^{i\langle h,v\rangle} - 1 - i\,\langle h,v\rangle\,\mathbb{1}_{B_V}(v),$$

then there exists a $C > 0$ such that $|K(h,v)| \leqslant C(\|h\|^2 \wedge 1)$ for all $h \in V$, which implies that $K(\cdot, v)$ is ξ_∞-integrable and hence by Lemma 3.3 in [24], we obtain

$$\lim_{t\to\infty} \int_V K(h,v)\xi_t(\mathrm{d}h) = \int_V K(h,v)\xi_\infty(\mathrm{d}h), \qquad \text{for all } v \in V. \tag{6.2.31}$$

If ν denotes the infinitely divisible probability measure with characteristics $(c_\infty, S_\infty, \xi_\infty)$, then it follows by (6.2.13), (6.2.30) and (6.2.31) that

$$\varphi_{\nu_t}(v) \to \varphi_\nu(v) \qquad \text{for all } v \in V,$$

which together with relative compactness of $\{\nu_t : t \geqslant 0\}$ implies by [36, Lemma VI.2.1] that $\{\nu_t : t \geqslant 0\}$ converges weakly in $\mathcal{M}(V)$. This finishes the proof of sufficiency by Lemma 6.2.4.
Necessity: suppose that $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ exists. Then by Lemma 6.2.4, $\{\nu_t\}$ converges weakly as $t \to \infty$. Then (6.2.13)-(6.2.16) follow by the compactness criterion of infinitely divisible probability measures in Hilbert spaces. $\square$

Example 6.2.12. (Continues Example 4.3.3) Let $(l_k)_{k\in\mathbb{N}}$ be a sequence of symmetric independent, real valued Lévy processes with characteristics $(0, 0, \mu_k)$, and L be the cylindrical Lévy process given by

$$L(t)u := \sum_{k=1}^\infty \langle e_k, u\rangle l_k(t).$$

Let the semigroup satisfy $T(t)e_k = e^{-\lambda_k t}e_k$ for all $t \geqslant 0$. Then the integral $\int_0^\infty T(s)\,\mathrm{d}L(s)$ exists if and only if (6.2.15) and (6.2.16) are satisfied. The independence of real valued processes (l_k) implies that the cylindrical Lévy measure is concentrated on the axes, and thus (6.2.15) and (6.2.16) can be seen to be equivalent to

$$\sum_{k=1}^{\infty} \int_0^{\infty} \int_{\mathbb{R}} \left(e^{-2\lambda_k s} |\beta|^2 \wedge 1 \right) \mu_k(\mathrm{d}\beta) \, \mathrm{d}t < \infty. \tag{6.2.32}$$

This example is studied in [40] and the above condition is shown to be necessary and sufficient for the existence of a unique stationary measure. It can be mentioned that in the specific case of cylindrical α-stable noise studied in [41], that is, $l_k = \sigma_k h_k$ where h_k are symmetric α-stable processes with Lévy measure $\rho(\mathrm{d}\beta) = \frac{1}{2}|\beta|^{-1-\alpha}$ and $\sigma \in l^{\frac{2\alpha}{2-\alpha}}$, unique stationary measure exists if and only if

$$\sum_{k=1}^{\infty} \frac{|\sigma_k|^\alpha}{\lambda_k} < \infty,$$

which is also the condition for stochastic integrability in our framework.

Example 6.2.13. (Continues Example 4.3.4) Let L be the canonical α-stable cylindrical Lévy process for $\alpha \in (0, 2)$ and the semigroup $(T(t))_{t \geqslant 0}$ be given by $T(t)e_k = e^{-\lambda_k t} e_k$. Using the same arguments as in Theorem 4.1 in [42], it can be shown that (6.2.15) and (6.2.16) are satisfied if and only if

$$\int_0^{\infty} \|T(s)\|_{\mathrm{HS}}^{\alpha} \, \mathrm{d}s < \infty, \tag{6.2.33}$$

which gives the condition of the existence of the integral $\int_0^\infty T(s)\, \mathrm{d}L(s)$.

The condition (6.2.13) for $\{c_t\}$ may be difficult to handle as $\{c_t\}$ is defined by (6.2.5) and the function a is not linear. If L is the genuine Lévy process with characteristics (b, Q, μ), then we get an explicit form for c_t. In this case, the cylindrical characteristics of L are given by (a, Q, μ) where

$$a(u^*) = \langle b, u^* \rangle + \int_U \langle u, u^* \rangle \left(\mathbb{1}_{B_{\mathbb{R}}}(\langle u, u^* \rangle) - \mathbb{1}_{B_U}(u)\right) \mu(\mathrm{d}u). \tag{6.2.34}$$

Then for every $v \in V$, we have by (6.2.5) and (6.2.34),

$$\begin{aligned}
\langle c_t, v \rangle &= \int_0^t a(B^*T^*(s)v)\,\mathrm{d}s \\
&\quad + \int_0^t \int_U \langle u, B^*T^*(s)v \rangle \left(1_{B_V}(T(s)Bu) - 1_{B_{\mathbb{R}}}(\langle u, B^*T^*(s)v \rangle)\right) \mu(\mathrm{d}u)\,\mathrm{d}s \\
&= \int_0^t \langle T(s)Bb, v \rangle\,\mathrm{d}s + \int_0^t \int_U \langle T(s)Bu, v \rangle \left(\mathbb{1}_{B_V}(T(s)Bu) - \mathbb{1}_{B_U}(u)\right) \mu(\mathrm{d}u)\,\mathrm{d}s.
\end{aligned}$$

As a consequence, we observe that Theorem 6.2.8 is equivalent to the well-known result from [16], that is, $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ exists if and only if the following conditions are satisfied:

(i) There exists

$$\lim_{t\to\infty} \left(\int_0^t T(s)Bb\,\mathrm{d}s + \int_0^t \int_U T(s)Bu \left(\mathbb{1}_{B_V}(T(s)Bu) - 1_{B_U}(u)\right) \mu(\mathrm{d}u)\,\mathrm{d}s \right);$$

(ii) $$\int_0^\infty \operatorname{tr}\left(T(s)BQB^*T^*(s)\right)\,\mathrm{d}s < \infty; \tag{6.2.35}$$

(iii) $$\int_0^\infty \int_U \left(\|T(s)Bu\|^2 \wedge 1\right) \mu(\mathrm{d}u)\,\mathrm{d}s < \infty. \tag{6.2.36}$$

The equivalence of (6.2.36) and the Conditions (6.2.15) and (6.2.16) can be obtained by noting that in this case μ is a genuine Lévy measure and consequently $\eta = (\mathrm{leb} \otimes \mu) \circ \chi_{[0,\infty)}^{-1}$ is also a genuine measure. By Lemma 6.2.11, the Conditions (6.2.15) and (6.2.16) are equivalent to the Condition that η is a Lévy measure which is equivalent to (6.2.36).

6.3 Case of stable semigroups

In the previous section we showed that the existence of the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ is sufficient for the existence of an invariant measure. We now show that if the semigroup is stable, then the existence of the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ is also necessary for the existence of an invariant measure. A semigroup $(T(t), t \geqslant 0)$ on V is called stable if $T(t)v \to 0$ as $t \to \infty$ for each $v \in V$. Similar to the case of genuine Lévy processes, we get the following result when the semigroup is stable, the proof of which is the same as in [16, Prop. 6.1].

Theorem 6.3.1. *If the semigroup $(T(t), t \geqslant 0)$ is stable, then there exists a stationary measure ν for the process (6.2.4) if and only if the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ exists; in this case ν is the distribution of $\int_0^\infty T(s)B\,\mathrm{d}L(s)$.*

Proof. If the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ exists and ν denotes the distribution of the random variable $\int_0^\infty T(s)B\,\mathrm{d}L(s)$, then by Lemma 6.2.4 and Lemma 6.2.7 ν is a stationary measure for (6.2.4). To prove the converse, let ν be a stationary measure for the process (6.2.4). We first show that $T_t\nu \to \delta_0$ weakly as $t \to \infty$. Since for all $v \in V$, $T(t)v \to 0$ as $t \to \infty$, for any continuous and bounded function $f\colon V \to V$, we have $f(T(t)v) \to f(0)$ as $t \to \infty$. By Lebesgue's theorem of dominated convergence, we get

$$\int_V f(v)(T_t\nu)(\mathrm{d}v) = \int_V f(T(t)v)\nu(\mathrm{d}v) \to f(0) = \int_V f(v)\delta_0(\mathrm{d}v).$$

This proves the claim. By definition, ν satisfies $\nu = T_t\nu * \nu_t$, for all $t \geqslant 0$. Then by [36, Theorem III.2.1, p. 58], $\{\nu_t\}$ is relatively compact and $\varphi_{\nu_t}(v) \to \varphi_\nu(v)$ for all $v \in V$. Therefore, by [36, Lemma VI.2.1, p. 153], ν_t converges weakly to ν, as $t \to \infty$. Finally by Lemma 6.2.4, the integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$ exists and ν is the distribution of the random variable $\int_0^\infty T(s)B\,\mathrm{d}L(s)$. □

Remark 6.3.2. The above result shows that in the case of stable semigroups, the stationary measure, if exists, is unique and is exactly the law of the random variable $\int_0^\infty T(s)B\,\mathrm{d}L(s)$.

Combining Theorem 6.3.1 with Theorem 6.2.8, we get the following result which gives the necessary and sufficient conditions for the existence of a unique stationary measure in the case of a stable semigroup.

Corollary 6.3.3. If the semigroup $(T(t), t \geqslant 0)$ is stable, then Conditions (6.2.13)-(6.2.16) of Theorem 6.2.8 are necessary and sufficient for the existence of a (unique) stationary measure for the process (6.2.4).

In general the conditions of Theorem 6.2.8 may be difficult to verify in practice. If the semigroup is exponentially stable, i.e. there exists $C > 1$ and $\lambda > 0$ such that $\|T(t)\| \leqslant Ce^{-\lambda t}$ for all $t \geqslant 0$, and L is a genuine Lévy process, then a sufficient condition (see [16, Theorem 6.7]) for the existence of stationary measure is that the Lévy measure satisfies the following simple condition

$$\int_U \log^+ \|u\| \mu(\mathrm{d}u) < \infty, \tag{6.3.1}$$

where $\log^+ x := \log x$ if $x \geqslant 1$ and 0 otherwise. This condition is also necessary if V is finite dimensional (see Theorem 4.3.17 in [4] and references therein) or if the semigroup $(T(t) : t \in (-\infty, \infty))$ is a group ([16, Prop. 6.8]) but in general is not necessary (see [15, Example 3.15]). In the case of a semigroup $(T(t) : t \geqslant 0)$ with spectral decomposition $T(t)e_k = e^{-\lambda_k} e_k$ (e.g. the heat semigroup) where the eigenvalues (λ_k) satisfy some mild conditions (see (6.3.4)), the following weaker condition

$$\int_U \sup_n \left(\frac{\log^+ |\langle u, e_n \rangle|}{\lambda_n} \right) \mu(\mathrm{d}u) < \infty,$$

is shown in [17] to be both necessary and sufficient for the existence of a stationary measure when L is a genuine Lévy process. In the next main result of this section, we generalise this condition for the case of symmetric cylindrical Lévy processes and give some examples satisfying this condition. In the rest of this section, we assume that $U = V$ with orthonormal basis $(e_k)_{k\in\mathbb{N}}$ and $B = \mathrm{Id}$. We also assume that A is a self-adjoint strictly negative operator with compact resolvent. Consequently, A has a purely point spectrum $(-\lambda_k)_{k=1}^{\infty}$, where

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots ; \qquad \lim_{k\to\infty} \lambda_k = \infty, \tag{6.3.2}$$

and there is an orthonormal basis e_k in V consisting of eigenvectors of A corresponding to the eigenvalues $-\lambda_k$. Then A is a generator of the C_0-semigroup $(T(t))_{t\geqslant 0}$ of bounded linear operators on V, given by the formula:

$$T(t)v = \sum_{k=1}^{\infty} e^{-\lambda_k t}\langle v, e_k\rangle e_k \quad \text{ for } v \in V. \tag{6.3.3}$$

Theorem 6.3.4. *Suppose that L has characteristics $(0, Q, \mu)$, where μ is symmetric, the semigroup is given by* (6.3.3) *and*

$$\sum_{k=1}^{\infty} \frac{e^{-\lambda_k T_0}}{\lambda_k} < \infty, \tag{6.3.4}$$

for some $T_0 > 0$. Then the following are equivalent:

(a) There exists a stationary measure for the process (6.2.4)*;*

(b) *(i)* $$\sup_{n\geqslant 1} \int_U \max_{1\leqslant k\leqslant n} \left(\frac{\log^+ |\langle u, e_k\rangle|}{\lambda_k} \right) \mu(\mathrm{d}u) < \infty; \tag{6.3.5}$$

(ii) $$\limsup_{m\to\infty} \sup_{n\geqslant m} \int_U \max_{m\leqslant k\leqslant n} \left(\frac{\log^+ |\langle u, e_k\rangle|}{\lambda_k} \right) \mu(\mathrm{d}u) = 0 \tag{6.3.6}$$

Proof. $(b) \Rightarrow (a)$. We show that the conditions in Theorem 6.2.8 are satisfied. By the stochastic integrability of the map $s \to T(s)$ in $[0, T_0]$, it follows by (2.4.4) that

$$\int_0^{T_0} \operatorname{tr}(T(s)QT^*(s))\,\mathrm{d}s < \infty. \tag{6.3.7}$$

Condition (6.3.4) implies

$$\begin{aligned}
\int_{T_0}^{\infty} \operatorname{tr}(T(s)QT^*(s))\,\mathrm{d}s &= \sum_{k=1}^{\infty} \int_{T_0}^{\infty} \langle T(s)QT^*(s)e_k, e_k\rangle \;\mathrm{d}s \\
&= \sum_{k=1}^{\infty} \langle Qe_k, e_k\rangle \int_{T_0}^{\infty} e^{-2\lambda_k s}\,\mathrm{d}s \\
&\leqslant \frac{1}{2}\|Q\|_{\mathrm{op}} \sum_{k=1}^{\infty} \frac{e^{-\lambda_k T_0}}{\lambda_k} \\
&< \infty,
\end{aligned}$$

which along with (6.3.7) implies that (6.2.14) is satisfied. We next show that (6.2.15) and (6.2.16) are satisfied. Again using the stochastic integrability of the map $s \to T(s)$ in $[0, T_0]$, we have

$$\limsup_{m\to\infty} \sup_{n\geqslant m} \int_0^{T_0} \int_U \left(\sum_{k=m}^{n} \langle u, T^*(s)e_k\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s = 0. \tag{6.3.8}$$

It follows by (2.3.4) that for any $c > 0$,

$$K_c := \sup_{\|u^*\|\leqslant c} \int_U \langle u, u^*\rangle^2 \wedge 1\,\mu(\mathrm{d}u) < \infty. \tag{6.3.9}$$

For $k, m, n \in \mathbb{N}$, $m \leqslant n$ and $s \geqslant T_0$, we define the following sets

$$C_k(s) := \left\{ u : |\langle u, e_k\rangle| < \exp\left(\frac{\lambda_k s}{2}\right) \right\}$$

$$B_{m,n} := \left\{ u : \sum_{k=m}^{n} \langle u, e_k \rangle^2 \leqslant 1 \right\}.$$

Using the spectral decomposition (6.3.3) of the semigroup, we have

$$\begin{aligned}
&\int_{T_0}^{\infty} \int_U \left(\sum_{k=m}^{n} \langle u, T^*(s) e_k \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\quad = \int_{T_0}^{\infty} \int_U \left(\sum_{k=m}^{n} e^{-2\lambda_k s} \langle u, e_k \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\quad = \int_{T_0}^{\infty} \int_{B_{m,n}} \left(\sum_{k=m}^{n} e^{-2\lambda_k s} \langle u, e_k \rangle^2 \right) \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\qquad + \int_{T_0}^{\infty} \int_{\cap_{j=m}^{n} C_j(s) \cap B_{m,n}^c} \left(\sum_{k=m}^{n} e^{-2\lambda_k s} \langle u, e_k \rangle^2 \right) \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\qquad + \int_{T_0}^{\infty} \int_{\cup_{j=m}^{n} C_j^c(s) \cap B_{m,n}^c} \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\quad =: I_{m,n}^1 + I_{m,n}^2 + I_{m,n}^3.
\end{aligned} \tag{6.3.10}$$

Since $B_{m,n} \subset \cap_{k=m}^{n} \{u : \langle u, e_k \rangle^2 \leqslant 1\}$, we have

$$\begin{aligned}
I_{m,n}^1 &\leqslant \sum_{k=m}^{n} \left(\int_{T_0}^{\infty} e^{-2\lambda_k s}\, \mathrm{d}s \right) \int_{\{u : \langle u, e_k \rangle^2 \leqslant 1\}} \langle u, e_k \rangle^2 \, \mu(\mathrm{d}u) \\
&\leqslant \sum_{k=m}^{n} \left(\frac{e^{-2\lambda_k T_0}}{2\lambda_k} \right) \sup_{\|u^*\| \leqslant 1} \int_U \langle u, u^* \rangle^2 \wedge 1 \, \mu(\mathrm{d}u) \\
&\leqslant K_1 \sum_{k=m}^{n} \left(\frac{e^{-\lambda_k T_0}}{\lambda_k} \right).
\end{aligned} \tag{6.3.11}$$

Using the definition of the sets $C_k(s)$, we have

$$I_{m,n}^2 \leqslant \int_{T_0}^{\infty} \int_{\cap_{j=m}^{n} C_j(s)} \sum_{k=m}^{n} e^{-\lambda_k s} \left(e^{-\lambda_k s} \langle u, e_k \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\, \mathrm{d}s$$

$$
\begin{aligned}
&\leqslant \sum_{k=m}^{n}\left(\int_{T_0}^{\infty} e^{-\lambda_k s}\,\mathrm{d}s\right)\int_U \langle u, e^{-\frac{\lambda_1}{2}T_0}e_k\rangle^2 \wedge 1\,\mu(\mathrm{d}u)\\
&\leqslant \sum_{k=m}^{n}\left(\frac{e^{-\lambda_k T_0}}{\lambda_k}\right)\sup_{\|u^*\|\leqslant c}\int_U \langle u, u^*\rangle^2 \wedge 1\,\mu(\mathrm{d}u)
\end{aligned}
$$

where $c := e^{-\frac{\lambda_1}{2}T_0}$. Therefore,

$$
I^2_{m,n} \leqslant K_c \sum_{k=m}^{n} \frac{e^{-\lambda_k T_0}}{\lambda_k}. \tag{6.3.12}
$$

Noting that $\cup_{k=m}^{n} C_k^c(s) = \left\{u : \max_{m\leqslant k\leqslant n}\left(\frac{2\log^+|\langle u, e_k\rangle|}{\lambda_k}\right) > s\right\}$, we have

$$
\begin{aligned}
I^3_{m,n} &\leqslant \int_{T_0}^{\infty}\int_{\cup_{k=m}^{n} C_k^c(s)} \mu(\mathrm{d}u)\,\mathrm{d}s\\
&\leqslant \int_0^{\infty}\int_{\left\{u:\max_{m\leqslant k\leqslant n}\left(\frac{2\log^+|\langle u,e_k\rangle|}{\lambda_k}\right)>s\right\}} \mu(\mathrm{d}u)\,\mathrm{d}s\\
&= \int_0^{\infty}\mu\left(\left\{u : \max_{m\leqslant k\leqslant n}\left(\frac{2\log^+|\langle u, e_k\rangle|}{\lambda_k}\right) > s\right\}\right)\mathrm{d}s\\
&= 2\int_U \max_{m\leqslant k\leqslant n}\frac{\log^+|\langle u, e_k\rangle|}{\lambda_k}\,\mu(\mathrm{d}u),
\end{aligned} \tag{6.3.13}
$$

where the last equality can be proved as follows using the Fubini's theorem,

$$
\begin{aligned}
&\int_0^{\infty}\mu\left(\left\{u : \max_{m\leqslant k\leqslant n}\left(\frac{2\log^+|\langle u, e_k\rangle|}{\lambda_k}\right) > s\right\}\right)\mathrm{d}s\\
&= \int_0^{\infty}\mu\circ\pi^{-1}_{e_m,\cdots,e_n}\left(\left\{\beta : \max_{m\leqslant k\leqslant n}\left(\frac{2\log^+|\beta_k|}{\lambda_k}\right) > s\right\}\right)\mathrm{d}s\\
&= \int_0^{\infty}\int_{\mathbb{R}^{n-m+1}} \mathbb{1}_{\left\{\max_{m\leqslant k\leqslant n}\left(\frac{2\log^+|\beta_k|}{\lambda_k}\right)>s\right\}}(\beta)\mu\circ\pi^{-1}_{e_m,\cdots,e_n}(\mathrm{d}\beta)\,\mathrm{d}s\\
&= \int_{\mathbb{R}^{n-m+1}}\int_0^{\infty} \mathbb{1}_{\left\{\max_{m\leqslant k\leqslant n}\left(\frac{2\log^+|\beta_k|}{\lambda_k}\right)>s\right\}}(\beta)\,\mathrm{d}s\,\mu\circ\pi^{-1}_{e_m,\cdots,e_n}(\mathrm{d}\beta)\\
&= 2\int_U \max_{m\leqslant k\leqslant n}\frac{\log^+|\langle u, e_k\rangle|}{\lambda_k}\,\mu(\mathrm{d}u).
\end{aligned} \tag{6.3.14}
$$

Hence substituting (6.3.11), (6.3.12) and (6.3.13) in (6.3.10) and using (6.3.4) and (6.3.6) it follows that

$$\limsup_{m\to\infty} \sup_{n\geqslant m} \int_T^\infty \int_U \left(\sum_{k=m}^{n} \langle u, T^*(s)e_k\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\, \mathrm{d}s = 0.$$

Taking $m = 1$ and using (6.3.5), above computations also imply that (6.2.15) is satisfied. $(a) \Rightarrow (b)$. Using the equality proved in (6.3.14), we have

$$\begin{aligned}
\int_U \max_{m\leqslant k\leqslant n} \frac{\log^+ |\langle u, e_k\rangle|}{\lambda_k} \mu(\mathrm{d}u) &= \int_0^\infty \mu \left\{ u : \max_{m\leqslant k\leqslant n} \left(\frac{\log^+ |\langle u, e_k\rangle|}{\lambda_k} \right) > s \right\} \mathrm{d}s \\
&= \int_0^\infty \int_{\left\{u : \max_{m\leqslant k\leqslant n} \left(\frac{\log^+ |\langle u, e_k\rangle|}{\lambda_k} \right) > s \right\}} \mu(\mathrm{d}u)\, \mathrm{d}s \\
&= \int_0^\infty \int_{\cup_{k=m}^{n} \left\{u : e^{-\lambda_k s} |\langle u, e_k\rangle| > 1\right\}} \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\leqslant \int_0^\infty \int_{\left\{u : \sum_{k=m}^{n} e^{-2\lambda_k s} |\langle u, e_k\rangle|^2 > 1\right\}} \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\leqslant \int_0^\infty \int_U \left(\sum_{k=m}^{n} \langle u, T^*(s)e_k\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\, \mathrm{d}s.
\end{aligned}$$

Consequently by Corollary 6.3.3 we obtain that conditions (6.3.5) and (6.3.6) are satisfied. □

Example 6.3.5. For the stochastic heat equation on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$ for some $d \in \mathbb{N}$, Condition (6.3.4) is satisfied by the eigenvalues of A, which in this case is given by the Laplace operator, that is, $A = \Delta$. Indeed, the eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ of Δ satisfy $ak^{2/d} \leqslant \lambda_k \leqslant bk^{2/d}$ for some constants $0 < a < b$. Then for $d = 1$, we have $\sum_{k=1}^\infty \frac{1}{\lambda_k} < \infty$ and for $d \geqslant 2$, we have $\sum_{k=1}^\infty \frac{1}{\lambda_k^p} < \infty$ if $p > \frac{d}{2}$.

Example 6.3.6. If L is a classical Lévy process (not necessarily symmetric) with classical characteristics (b, Q, μ), and the semigroup is given by (6.3.3) and (6.3.4) holds, then by Theorem 1 in [17] the necessary and sufficient condition for the existence of stationary measure for the process (6.2.4) is

$$\int_U \sup_n \left(\frac{\log^+ |\langle u, e_n \rangle|}{\lambda_n} \right) \mu(\mathrm{d}u) < \infty. \tag{6.3.15}$$

In this case (6.3.5) and (6.3.6) are equivalent to (6.3.15). Since μ is a genuine Lévy measure, by monotone convergence theroem,

$$\sup_{n \geqslant m} \int_U \max_{m \leqslant k \leqslant n} \left(\frac{\log^+ |\langle u, e_k \rangle|}{\lambda_k} \right) \mu(\mathrm{d}u) = \int_U \sup_{n \geqslant m} \left(\frac{\log^+ |\langle u, e_n \rangle|}{\lambda_n} \right) \mu(\mathrm{d}u),$$

which shows the equivalence of (6.3.5) and (6.3.15) by taking $m = 1$. For each $u \in U$,

$$\sup_{n \geqslant m} \frac{\log^+ |\langle u, e_n \rangle|}{\lambda_n} \leqslant \sup_{n \geqslant m} \frac{\log^+ \|u\|}{\lambda_n} \leqslant \frac{\log^+ \|u\|}{\lambda_m} \to 0 \text{ as } m \to \infty.$$

An application of Lebesgue's theorem together with (6.3.15) implies (6.3.6).

Example 6.3.7. (continues Example 6.2.12) Since the Lévy measure is concentrated on the axes, (6.3.5) and (6.3.6) are satisfied if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_{\mathbb{R}} \log^+ |\beta| \mu_k(\mathrm{d}\beta) < \infty. \tag{6.3.16}$$

Suppose $l_k = \sigma_k l_k'$, where l_k' are symmetric and identically distributed with characteristics $(0, 0, \rho)$. Then the above condition gives

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_{\mathbb{R}} \log^+ |\sigma_k \beta| \, \rho(\mathrm{d}\beta) < \infty. \tag{6.3.17}$$

This condition is satisfied for example, when $\sigma \in l^\infty$, $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$ and

$$\int_1^{\infty} \log \beta \, \rho(\mathrm{d}\beta) < \infty, \tag{6.3.18}$$

which gives the result of [40].

Example 6.3.8. (continues Example 6.2.13) We now apply Theorem 6.3.4 to prove the existence of invariant measure in the case of canonical α-stable process. We assume that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty, \tag{6.3.19}$$

which implies (6.3.4) is satisfied. By Theorem 6.3.4 the existence of stationary measure is guaranteed if (6.3.5) and (6.3.6) are satisfied. For each $m, n \in \mathbb{N}$, $n \geqslant m$, we have

$$\begin{aligned}
&\int_U \max_{m \leqslant k \leqslant n} \left(\frac{\log^+ |\langle u, e_k \rangle|}{\lambda_k} \right) \mu(\mathrm{d}u) \\
&\quad \leqslant \int_U \left(\sum_{k=m}^{n} \frac{\log^+ |\langle u, e_k \rangle|}{\lambda_k} \right) \mu(\mathrm{d}u) \\
&\quad = \int_{\mathbb{R}^{n-m+1}} \left(\sum_{k=m}^{n} \frac{\log^+ |\beta_k|}{\lambda_k} \right) \mu \circ \pi^{-1}_{e_m, \cdots, e_n}(\mathrm{d}\beta) \\
&\quad = \frac{\alpha}{c_\alpha} \int_{S(\mathbb{R}^{n-m+1})} \int_0^\infty \left(\sum_{k=m}^{n} \frac{\log^+ |r\xi_k|}{\lambda_k} \right) \frac{1}{r^{1+\alpha}} \,\mathrm{d}r\, \lambda_{n-m+1}(\mathrm{d}\xi) \\
&\quad = \frac{\alpha}{c_\alpha} \int_{S(\mathbb{R}^{n-m+1})} \sum_{k=m}^{n} \frac{1}{\lambda_k} \int_0^\infty \left(\log^+ |r\xi_k| \right) \frac{1}{r^{1+\alpha}} \,\mathrm{d}r\, \lambda_{n-m+1}(\mathrm{d}\xi)
\end{aligned} \tag{6.3.20}$$

Using Fubini's theorem, we can compute

$$\begin{aligned}
\int_0^\infty \left(\log^+ |r\xi_k| \right) \frac{1}{r^{1+\alpha}} \,\mathrm{d}r &= \int_0^\infty \left(\int_0^\infty \mathbb{1}_{\{1 \leqslant u < r|\xi_k|\}} \frac{\mathrm{d}u}{u} \right) \frac{1}{r^{1+\alpha}} \,\mathrm{d}r \\
&= \int_0^\infty \left(\int_0^\infty \mathbb{1}_{\{1 \leqslant u < r|\xi_k|\}} \right) \frac{1}{r^{1+\alpha}} \,\mathrm{d}r \frac{\mathrm{d}u}{u} \\
&= \int_1^\infty \int_{\frac{u}{|\xi_k|}}^\infty \frac{1}{r^{1+\alpha}} \,\mathrm{d}r \frac{\mathrm{d}u}{u} \\
&= \int_1^\infty \frac{|\xi_k|^\alpha}{\alpha u^{1+\alpha}} \,\mathrm{d}u \\
&= \frac{|\xi_k|^\alpha}{\alpha^2}.
\end{aligned} \tag{6.3.21}$$

Define $\lambda^1_{n-m+1} := \frac{1}{r_{n-m+1}}\lambda_{n-m+1}$ where $r_{n-m+1} := \lambda_{n-m+1}\left(S(\mathbb{R}^{n-m+1})\right)$. Substituting the value of the integral computed in (6.3.21) into (6.3.20) and applying Jensen's inequality to the concave function $\beta \mapsto \beta^{\alpha/2}$, it follows that

$$\begin{aligned}
&\int_U \max_{m\leqslant k\leqslant n}\left(\frac{\log^+|\langle u, e_k\rangle|}{\lambda_k}\right)\mu(\mathrm{d}u) \\
&\qquad \leqslant \frac{r_{n-m+1}}{\alpha c_\alpha}\sum_{k=m}^{n}\frac{1}{\lambda_k}\left(\int_{S(\mathbb{R}^{n-m+1})}\xi_k^2\,\lambda^1_{n-m+1}(\mathrm{d}\xi)\right)^{\alpha/2} \\
&\qquad = \frac{r_{n-m+1}}{\alpha c_\alpha (n-m+1)^{\alpha/2}}\sum_{k=m}^{n}\frac{1}{\lambda_k}
\end{aligned} \tag{6.3.22}$$

Defining $d_{n-m} := \frac{r_{n-m+1}}{(n-m+1)^{\alpha/2}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-m+1+\alpha}{2}\right)}{(n-m+1)^{\alpha/2}\Gamma\left(\frac{n-m+1}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)}$ and using the fact that $\frac{\Gamma(x+\beta)}{\Gamma(x)x^\beta} \to 1$ as $x \to \infty$, we conclude that $d_{n-m} \to 1$ as $m, n \to \infty$. Consequently, both (6.3.6) and (6.3.5) follow from (6.3.19) and (6.3.22) respectively.

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